Intermediate Macroeconomics

ECO 301 — Lecture Notes

Feda Mohammadi

Professor: Dr. Nabila Rahman

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Textbook

Charles I. Jones, Macroeconomics, 5th Edition

These notes emphasize models, equations, and worked examples.

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PART I

Preliminaries

Introduction to Macroeconomics

1.1 What Is Macroeconomics?

Definition

Macroeconomics is the study of collections of people and firms and how their interactions through markets determine the *aggregate* performance of the economy — total production, total income, unemployment, inflation, and growth.

Core questions macroeconomics asks:

How rich is a country?	Why do some countries grow faster?	Why do recessions happen?
Why does inflation rise?	Why is unemployment sometimes high?	What drives interest rates?

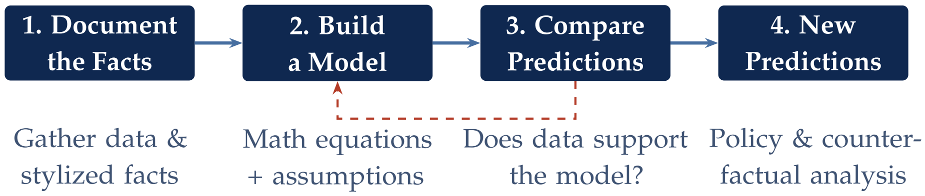
Key Idea: Macroeconomics studies how output, employment, wages, interest rates, and prices *interact* — a change in one ripples through the others.

Long Run vs. Short Run — The Big Split

	Long Run	**Short Run**
Time horizon	Decades	Months to a few years
Key question	Why do living standards rise?	Why do recessions occur?
Main variable	GDP per capita growth	Fluctuations around trend
Prices	Flexible (adjust fully)	Sticky (slow to adjust)

1.2 The Four-Step Method

Macroeconomics follows a systematic, scientific method:



The dashed feedback arrow means: if predictions fail, revise the model.

1.3 The Structure of an Economic Model

Definition

A **model** is a simplified system of mathematical equations and assumptions that isolates the most important forces at work in the economy.

Term	Definition
Endogenous variable	Determined *inside* the model (unknowns we solve for).
Exogenous variable	Determined *outside* the model; changes in a pre-specified way.
Parameter	A fixed structural number chosen by the model-builder ($\bar{}$ notation).
Equilibrium	The solution where all markets clear (supply = demand).

1.4 The Labor Market Model

This full model illustrates how endogenous variables are solved from a system of equations.

1.4.1 General Parameterized Version

Key Equation

$$L^s = \bar{a}\,w + \bar{\ell} \qquad \text{(Labor Supply)}$$
$$L^d = \bar{f} - w \qquad \text{(Labor Demand)}$$
$$L^s = L^d \qquad \text{(Equilibrium Condition)}$$

Symbol	Meaning	Type	Notes
w	Wage	**Endogenous**	Solved from the model
L	Employment	**Endogenous**	Solved from the model
$\bar{a}$	Slope of labor supply	**Parameter**	How responsive workers are to wage
$\bar{\ell}$	Labor supplied at $w = 0$	**Parameter**	Reflects intrinsic work preference
$\bar{f}$	Labor demanded at $w = 0$	**Parameter**	Reflects firm capacity

1.4.2 Equilibrium Solution

Set $L^s = L^d$ and solve:

$$\bar{a}\,w + \bar{\ell} = \bar{f} - w \quad \Longrightarrow \quad (\bar{a}+1)\,w = \bar{f} - \bar{\ell}$$

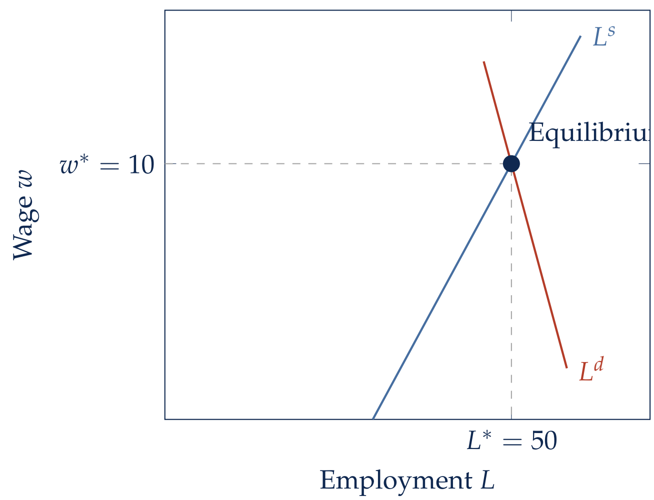
Key Equation

$$w^* = \frac{\bar{f} - \bar{\ell}}{\bar{a}+1} \qquad L^* = \bar{a}\left(\frac{\bar{f} - \bar{\ell}}{\bar{a}+1}\right) + \bar{\ell}$$

1.4.3 Comparative Statics

Shock	Effect on w^*	Effect on L^*	Economic Intuition
$\bar{\ell}\uparrow$ **(workers want to work more)**	$\downarrow$	$\uparrow$	Supply $\uparrow \Rightarrow$ wage falls, qty rises
$\bar{f}\uparrow$ **(firms want more labor)**	$\uparrow$	$\uparrow$	Demand $\uparrow \Rightarrow$ both rise
$\bar{a}\uparrow$ **(workers more wage-sensitive)**	$\downarrow$	ambiguous	Flatter supply curve

1.4.4 Supply–Demand Diagram



1.5 Worked Examples

Example 1 — Numeric Labor Market

Given: $L^s = 2w + 30, \quad L^d = 60 - w$
Endogenous variables: w and L
Solve:

$$2w + 30 = 60 - w \implies 3w = 30 \implies \boxed{w^* = 10}$$

$$L^* = 2(10) + 30 = \boxed{50}$$

Example 2 — Comparative Statics: $\bar{\ell}$ Increases

Given: General model with equilibrium $w^* = \dfrac{\bar{f} - \bar{\ell}}{\bar{a} + 1}$

If $\bar{\ell} \uparrow$ (workers want to work more even at wage $= 0$):

- w^* decreases — more supply pushes the wage down.
- L^* increases — more workers employed at the lower wage.

Diagram: Labor supply curve shifts *right*.
New equilibrium has lower wage and higher employment.

Chapter 1 — Summary

Key Takeaways

1. Macroeconomics studies economy-wide aggregates (output, unemployment, inflation, growth).
2. It uses a four-step method: document facts $\to$ build model $\to$ test predictions $\to$ new predictions.
3. A model has **exogenous inputs** (parameters + exogenous variables) and produces **endogenous outputs**.
4. Equilibrium solves supply $=$ demand simultaneously for all endogenous variables.
5. Comparative statics: change one parameter, re-solve, and track the effect on endogenous variables.

Measuring the Macroeconomy

2.1 What Is GDP?

Definition

Gross Domestic Product (GDP) is the market value of the *final* goods and services *produced within an economy* during a given period of time.

Four important words:

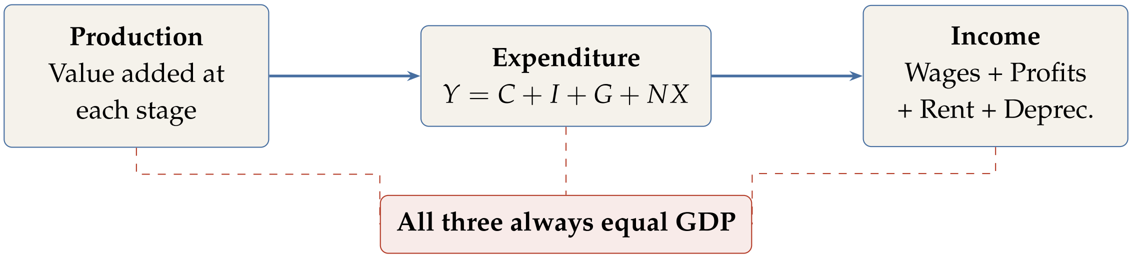
Word/Phrase	Why it matters
Market value	Prices let us add cars + haircuts + software into one number.
Final goods	Intermediate goods excluded — prevents double counting.
Produced within	Location of production, not ownership. A Toyota built in Kentucky is U.S. GDP.
Given period	GDP is a *flow* (per year or quarter), not a stock.

2.2 The Three Approaches to GDP

Key Equation

$$\text{Production} = \text{Expenditure} = \text{Income}$$

Every dollar of output produced is purchased by someone (expenditure) and becomes income for someone (wages, profits, rent).



2.3 The Expenditure Approach

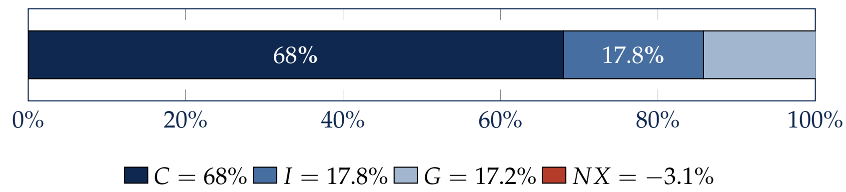
Key Equation

$$Y = C + I + G + NX \qquad \text{(National Income Identity)}$$

where $NX = X - M$ (Exports minus Imports)

U.S. GDP Breakdown — 2018 Source: BEA

Component	Total ($B)	% of GDP	Per person ($)
GDP	20,500	100.0	62,600
Consumption (C)	13,950	68.0	42,600
Motor vehicles & parts	510	2.5	1,550
Food	390	1.9	1,210
Housing	2,560	12.5	7,820
Medical care	2,380	11.6	7,250
Investment (I)	3,650	17.8	11,150
Structures (nonresidential)	640	3.1	1,950
Equipment	1,240	6.0	3,780
Intellectual property products	930	4.5	2,830
Residential	800	3.9	2,430
Government purchases (G)	3,520	17.2	10,760
National defense	780	3.8	2,380
Net exports (NX)	−630	−3.1	−1,910
Exports	2,530	12.3	7,730
Imports	3,160	15.4	9,640



Government purchases (G) ≠ Government spending. G includes only purchases of goods and services. It excludes *transfer payments* (Social Security, Medicare, unemployment insurance) — these redistribute income but do not directly produce output.

2.4 The Income Approach

U.S. GDP by Income — 2018 Source: BEA

Income Category	Total ($B)	% of GDP	Per person ($)
GDP	20,500	100.0	62,600
Compensation of employees	10,870	53.0	33,190
Wages and salaries	8,850	43.2	27,020
Benefits	2,020	9.9	6,170
Taxes less subsidies on production	1,360	6.6	4,160
Net operating surplus (profits)	5,900	28.8	18,000
Depreciation of fixed capital	3,270	16.0	10,000

Labor vs. Capital split:

Stylized Fact: Labor's share of GDP $\approx \frac{2}{3}$ and has been broadly stable since the 1950s — though there is a mild downward trend since ∼1970. **Capital's share** $\approx \frac{1}{3}$. This ratio is used repeatedly in growth models (Chapter 4–5).

2.5 The Production Approach & Value Added

Definition

$$\text{Value Added} = \text{Revenue} - \text{Cost of Intermediate Inputs}$$

GDP counts only *value added* at each stage — not the gross revenue at every stage — to avoid double counting.

Example: Steel → Automobile

Steel producer
Revenue: $10M
Inputs: $0
VA: $10M

→

Auto producer
Revenue: $100M
Inputs: $10M (steel)
VA: $90M

→

GDP contribution
$= \$10M + \$90M$
$= \$100M$

What counts and what does not:

Included in GDP		NOT Included
Final goods & services	produced domestically	Intermediate goods
New production this period	market transactions	Used goods
Gov't purchases of goods	services only	Transfer payments
		Financial transactions
		Household production
		Underground economy

2.6 Nominal vs. Real GDP

Key Equation

$$\underbrace{\text{Nominal GDP}}_{\text{current prices}} = \underbrace{P}_{\text{price level}} \times \underbrace{\text{Real GDP}}_{\text{constant prices}}$$

	Nominal GDP	**Real GDP**
Prices used	Current year prices	Fixed base-year prices
What changes	Prices *and* quantities	Quantities only
Useful for	Measuring total dollar value	Measuring actual growth
Problem	Inflation distorts comparisons	Must choose a base year

Worked Numerical Example (apples & computers):

	2025			**2026**		
Good	P	Q	PQ	P	Q	PQ
Apples	$1	500	$500	$2	500	$1,000
Computers	$900	5	$4,500	$1,000	5	$5,000
Nominal GDP			**$5,000**			**$6,000**

Quantities are *identical* in 2025 and 2026. The entire $1,000 rise in nominal GDP is due to higher prices — **real GDP is unchanged**. This is why we need real GDP.

2027 example (quantities increase):

	2026	**2027**
Apples: price	$2	$3
Apples: quantity	500	550
Computers: price	$1,000	$1,000
Computers: quantity	5	6
Nominal GDP	$6,000	$7,650

2.7 Price Indexes: Laspeyres, Paasche, Chain

Laspeyres

Use *base-year* prices

$$g_L = \frac{\sum P_{\text{old}}\, Q_{\text{new}} - \sum P_{\text{old}}\, Q_{\text{old}}}{\sum P_{\text{old}}\, Q_{\text{old}}}$$

Paasche

Use *current-year* prices

$$g_P = \frac{\sum P_{\text{new}}\, Q_{\text{new}} - \sum P_{\text{new}}\, Q_{\text{old}}}{\sum P_{\text{new}}\, Q_{\text{old}}}$$

Chain-Weighted

Average of both:

$$g_{\text{chain}} = \frac{g_L + g_P}{2}$$

Used by BEA (U.S.)

Numerical Results — 2026 to 2027

$$\begin{aligned}
\text{Laspeyres (2026 prices):}\quad & \text{RGDP}_{2026} = 2(500) + 1000(5) = 6{,}000 \\
& \text{RGDP}_{2027} = 2(550) + 1000(6) = 7{,}100 \\
& g_L = \frac{7{,}100 - 6{,}000}{6{,}000} = \mathbf{18.3}\% \\
\text{Paasche (2027 prices):}\quad & \text{RGDP}_{2027} = 3(550) + 1000(6) = 7{,}650 \\
& \text{RGDP}_{2026} = 3(500) + 1000(5) = 6{,}500 \\
& g_P = \frac{7{,}650 - 6{,}500}{6{,}500} = \mathbf{17.7}\% \\
\text{Chain-weighted:}\quad & g_{\text{chain}} = \frac{18.3 + 17.7}{2} = \mathbf{18.0}\%
\end{aligned}$$

2.8 The GDP Deflator & Inflation

Key Equation

$$\text{GDP Deflator} = \frac{\text{Nominal GDP}}{\text{Real GDP}}$$

$$\underbrace{\%\Delta\,\text{Nominal GDP}}_{\approx 27.5\%} \approx \underbrace{\%\Delta\, P}_{\text{inflation}} + \underbrace{\%\Delta\,\text{Real GDP}}_{\approx 18.0\%}$$

$$\Longrightarrow \quad \pi \approx 27.5\% - 18.0\% = \mathbf{9.5}\%$$

Index	Real GDP growth	Implied inflation
Laspeyres	18.3%	7.9%
Paasche	17.7%	8.5%
Chain-weighted (Fisher)	18.0%	**9.5%**

Why chain-weighting? When relative prices shift significantly over time (e.g., computers getting dramatically cheaper), fixing prices at either endpoint creates large distortions. Chain-weighting updates prices year-by-year, eliminating this bias. The U.S. BEA adopted chain-weighting in the late 1990s.
Caution: Chain-weighted real components do *not* add up to chain-weighted real GDP. Use *nominal shares* when computing component shares of GDP.

2.9 Comparing GDP Across Countries

Two steps are required to compare GDP across countries:

1. **Convert to a common currency** using exchange rates.
2. **Adjust for price level differences** using Purchasing Power Parity (PPP).

Key Equation

$$\text{Real GDP}^{\text{U.S. prices}}_{\text{China}} = \left(\frac{P_{\text{U.S.}}}{P_{\text{China}}}\right) \times \text{Nominal GDP}^{\text{dollars}}_{\text{China}}$$

China example (2017):

Step	**Calculation**
Nominal GDP (China, \$)	$\approx$ \$11.1 trillion (converted at exchange rate)
Price level ratio $P_{\text{U.S.}}/P_{\text{China}}$	$\approx 1/0.625$ (goods in China cost 62.5% of U.S. prices)
Real GDP at U.S. prices	\$11.1/0.625 $\approx$ \$17.8 trillion
U.S. GDP	$\approx$ \$17.7 trillion
Conclusion	At common prices, China $\approx$ U.S. in total GDP (not 40% smaller as exchange rates suggest).

General rule: Rich countries have higher price levels. Comparing with exchange rates *overstates* the gap between rich and poor countries. PPP-adjusted comparisons (Penn World Tables) give more accurate pictures of living standards.

2.10 Worked Examples

Example 1 — What Counts as GDP?

(a) You pay \$5,000 in college tuition. **GDP rises by \$5,000** — final service.
(b) You buy a used car for \$2,500. **GDP unchanged** — already counted when produced.
(c) Government spends \$100M on a dam. **GDP rises by \$100M** — government purchase (G).
(d) A computer firm buys parts for \$1M, sells computers for \$2M. **GDP rises by \$1M** — only value added counts.
(e) Government raises unemployment benefits by \$100M. **GDP unchanged** — transfer payment, not a purchase of goods/services.
(f) U.S. airline imports \$50M of Airbus jets. **GDP unchanged** — I rises by \$50M but M rises by \$50M, so NX falls by \$50M; net effect = 0.

Example 2 — Orange & Boomerang Economy

Data: Oranges and boomerangs in 2020 and 2021.

	2020	**2021**	**% change**
Q_{oranges}	100	105	+5.0%
$Q_{\text{boomerangs}}$	20	22	+10.0%
P_{oranges}	\$1.00	\$1.10	+10.0%
$P_{\text{boomerangs}}$	\$3.00	\$3.10	+3.3%
Nominal GDP	\$160	\$183.7	+14.8%
Real GDP (2020 prices)	\$160	\$171	+6.9%
Real GDP (2021 prices)	\$172	\$183.7	+6.8%
Chain-weighted growth	$(6.9\% + 6.8\%)/2 = \mathbf{6.84}\%$		

Inflation (GDP deflator):

$$\pi \approx \%\Delta\,\text{Nominal GDP} - \%\Delta\,\text{Real GDP} = 14.8\% - 6.84\% \approx \mathbf{7.9}\%$$

Chapter 2 — Summary

Key Takeaways

1. GDP = market value of final goods and services produced domestically in a period.
2. **Three approaches always agree:** Production = Expenditure ($Y = C + I + G + NX$) = Income.
3. G excludes transfer payments. Imports are subtracted because C, I, G already include them.
4. Labor $\approx \frac{2}{3}$ of GDP; Capital $\approx \frac{1}{3}$, a key stylized fact used in all growth models.
5. **Nominal GDP** uses current prices; **Real GDP** uses fixed prices, only real GDP measures actual output growth.
6. **Chain-weighting** averages Laspeyres and Paasche growth rates to reduce index bias.
7. GDP deflator $=$ Nominal GDP/Real GDP; its growth rate is the inflation rate.
8. Cross-country comparisons require PPP adjustment, exchange rates overstate income gaps.

PART II

The Long Run

An Overview of Long-Run Economic Growth

3.1 The Big Picture

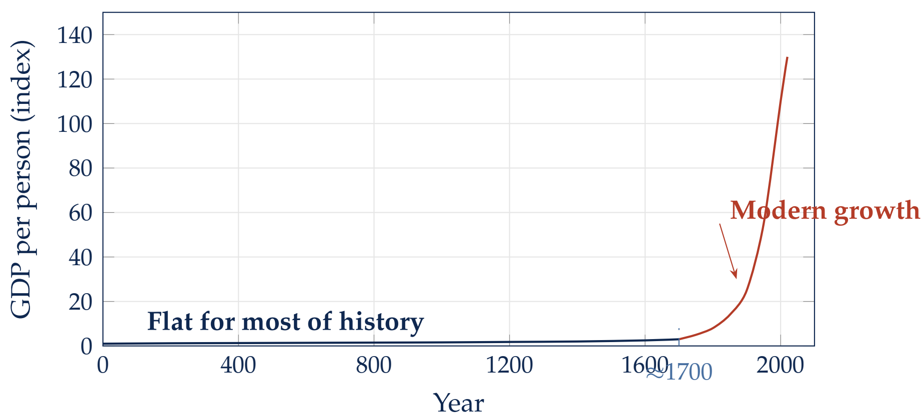
Definition

Economic growth is the sustained increase in real GDP per capita over long periods of time. It is the single most powerful force for improving human living standards.

The U.S. in 1870 vs. today: In 1870, life expectancy was under 50, 1 in 10 infants died before age 1, over 90% of households had no electricity, refrigerator, or telephone, and fewer than 10% of young adults had a high school diploma. That description fits the United States — not Ethiopia or Haiti today. Per capita GDP: \$3,600 (1870) $\rightarrow$ \$61,000 (2018). A **15-fold increase** in 148 years.

3.2 Growth over the Very Long Run

Key fact: Sustained growth is a *very* recent phenomenon.



Stylized "hockey stick": GDP per person over 2,000 years. Growth is essentially zero for centuries, then explodes.

Country patterns from Figure 3.1 (Jones):

Country	Pattern
United Kingdom	First country to experience sustained modern growth (∼1700s).
United States	Follows UK; eventually becomes richest. Steady ∼2%/yr since 1870.
Japan	Rapid growth after WWII (∼6%/yr, 1950–1990); converged to ∼3/4 U.S. level.
Argentina	Strong growth 1900–1980; stagnation after 1980.
China	Growth takes off after 1978; > 8%/yr recently; now ∼1/4 U.S. level.
Ghana	Weak long-run growth; ≈ 1/15 of U.S. income today.

The Great Divergence

Before 1700, living standards across countries differed by at most a factor of 2–3. Today they differ by a factor of **more than 100**. Countries that began modern growth earlier had more time for incomes to compound — this widening gap is called the **Great Divergence**.

3.3 Mathematical Tools of Growth

3.3.1 The Growth Rate

Key Equation

One-period growth rate:

$$g_t = \frac{y_{t+1} - y_t}{y_t}$$

Constant growth rule (if g is constant every period):

$$y_t = y_0\,(1+g)^t \tag{3.1}$$

Average annual growth rate between year 0 and year t:

$$g = \left(\frac{y_t}{y_0}\right)^{1/t} - 1 \tag{3.2}$$

U.S. example: $y_0 = \$3{,}600$ (1870), $y_t = \$61{,}000$ (2018), $t = 148$:

$$g = \left(\frac{61{,}000}{3{,}600}\right)^{1/148} - 1 = 0.0193 \approx \boxed{2\% \text{ per year}}$$

3.3.2 The Rule of 70

Key Equation

If a variable grows at g% per year, it **doubles** approximately every:

$$\text{Years to double} \;\approx\; \frac{70}{g}$$

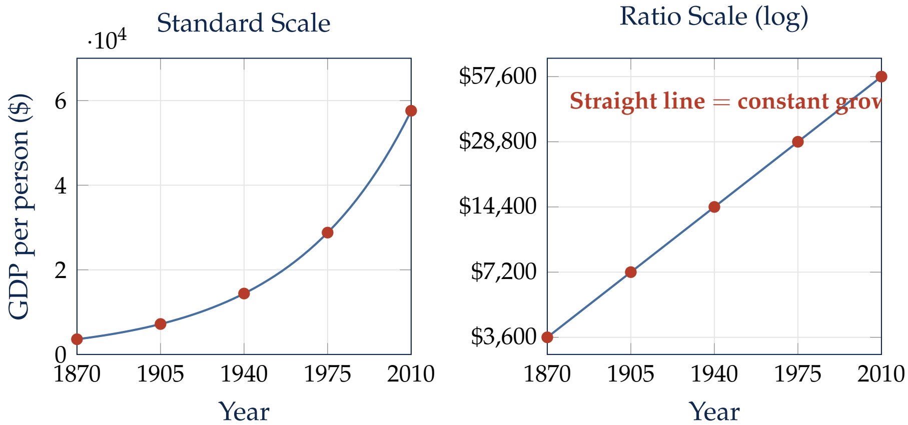
Growth Rate	Years to Double	Example
1% / year	70 years	Slow-growing economy
2% / year	35 years	U.S. long-run average
4% / year	17.5 years	Fast-growing developing country
6% / year	11.7 years	China/Korea at peak
7% / year	10 years	Income $\times 8$ in 30 years

Why this matters: At 2%/yr, it takes 35 years to double income. At 6%/yr, income doubles in 12 years, quadruples in 24, and grows $\times 8$ in 36 years. A seemingly small difference in growth rates leads to enormous differences over a generation.

3.3.3 The Ratio (Log) Scale

Definition

A **ratio scale** (logarithmic scale) labels the vertical axis so that equal vertical distances represent equal *percentage* changes (doublings). A variable growing at a **constant rate** appears as a **straight line** on a ratio scale.



3.4 Growth Rules (we use it a lot)

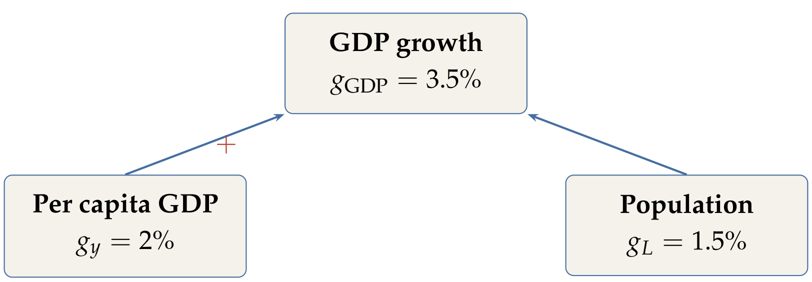
Three Growth Rate Rules

Let g_x and g_y be the growth rates of variables x and y.

Rule	If...	Then...
1	$z = x/y$	$g_z = g_x - g_y$
2	$z = x \times y$	$g_z = g_x + g_y$
3	$z = x^a$	$g_z = a \cdot g_x$

Key application — GDP, population, and per capita GDP:

$$\text{GDP} = \underbrace{y}_{\text{per capita}} \times \underbrace{L}_{\text{population}} \implies g_{\text{GDP}} = g_y + g_L$$



Examples using the three rules:

Expression	Rule	Growth rate
$z = xy$, $g_x = 4\%$, $g_y = 2\%$	Rule 2	$g_z = 6\%$
$z = x/y$, $g_x = 4\%$, $g_y = 2\%$	Rule 1	$g_z = 2\%$
$z = y/x$, $g_x = 4\%$, $g_y = 2\%$	Rule 1	$g_z = -2\%$
$z = x^{1/2}y^{1/2}$	Rules 2+3	$g_z = \frac{1}{2}(4\%) + \frac{1}{2}(2\%) = 3\%$
$z = (x/y)^2$	Rules 1+3	$g_z = 2(4\% - 2\%) = 4\%$
$z = x^{-1/3}y^{2/3}$	Rules 1+3	$g_z = -\frac{1}{3}(4\%) + \frac{2}{3}(2\%) = 0\%$

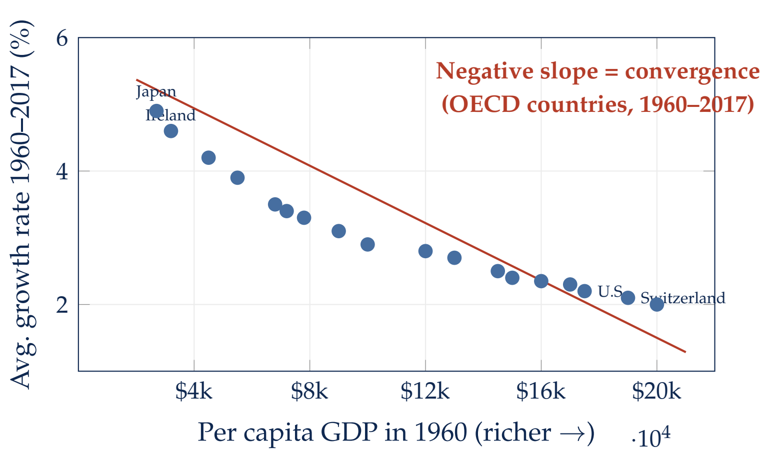
3.5 Modern Growth Around the World

3.5.1 Convergence

Definition

Convergence is the tendency for poorer countries that are below their long-run steady state to grow faster than richer countries, gradually closing the income gap.

Observed clearly within the OECD after WWII (Germany, Japan, Ireland, Spain). Not observed across the full world sample — many poor countries have not converged.



3.5.2 Cross-Country Income Facts

Country / Group	Per capita GDP (rel. to U.S.)	Notes
U.S., Ireland, Norway	≈ 1	Richest in world
U.K., Japan, Germany	$\approx 3/4$	Converged post-WWII
Argentina, China	$\approx 1/4$–$1/3$	Middle income
India	$\approx 1/8$	Rapid recent growth
Ghana	$\approx 1/15$	Slow long-run growth
Poorest countries	$< 1/64$	Large income spread

Scale of global income differences: Before 1700, incomes across countries differed by at most $2\times$–$3\times$. Today they differ by more than **100**$\times$. Growth rates from 1960–2017 range from -2% to $+7\%$ per year across all countries.

3.6 Costs and Benefits of Economic Growth

Benefits	Costs
Higher incomes & living standards	**Environmental damage (pollution, CO_2)**
Longer life expectancy	**Resource depletion**
Better health outcomes	**Rising income inequality**
More goods, services, technology	**Job displacement from technology**
Poverty reduction (global)	**Social disruption and urbanization**

Consensus view: Benefits outweigh costs, especially in poor countries where the main problem is *too little* growth, not too much. Environmental damage often peaks at middle income levels and then declines as countries grow richer (the "environmental Kuznets curve").

3.7 Worked Examples

Example 1 — Ethiopia Growth Projections

$y_0 = \$1{,}600$ (2017), $t = 33$ years (to 2050). Apply: $y_t = y_0(1+g)^t$

g	$(1+g)^{33}$	y_{2050}	Comparison
1%	1.388	\$2,221	Still very poor
2%	1.923	\$3,077	Still very poor
4%	3.652	\$5,843	Approaching middle income
6%	6.843	\$10,949	Near Mexico's 2017 level (\$17,000)

Key insight: Growth rate differences that look small (1% vs. 6%) produce wildly different outcomes over just 33 years.

Example 2 — Growth Rate Rules Applied

Given: $g_x = 4\%$, $g_y = 2\%$. Find g_z:

(a) $z = xy \quad\Rightarrow\quad g_z = 4 + 2 = \boxed{6\%}$
(b) $z = x/y \quad\Rightarrow\quad g_z = 4 - 2 = \boxed{2\%}$
(c) $z = y/x \quad\Rightarrow\quad g_z = 2 - 4 = \boxed{-2\%}$
(d) $z = x^{1/2}y^{1/2} \quad\Rightarrow\quad g_z = \frac{1}{2}(4) + \frac{1}{2}(2) = \boxed{3\%}$
(e) $z = (x/y)^2 \quad\Rightarrow\quad g_z = 2(4-2) = \boxed{4\%}$
(f) $z = x^{-1/3}y^{2/3} \quad\Rightarrow\quad g_z = -\frac{1}{3}(4) + \frac{2}{3}(2) = \boxed{0\%}$

Example 3 — Computing Average Growth Rate

U.S. growth 1870–2018: $y_0 = \$3{,}600$, $y_t = \$61{,}000$, $t = 148$

$$g = \left(\frac{61{,}000}{3{,}600}\right)^{1/148} - 1 = (16.94)^{0.00676} - 1 = 0.0193 \approx \boxed{2\% \text{ per year}}$$

U.S. growth 1870–1929: $y_0 = \$3{,}629$, $y_t = \$9{,}827$, $t = 59$

$$g = \left(\frac{9{,}827}{3{,}629}\right)^{1/59} - 1 \approx \boxed{1.69\% \text{ per year}}$$

U.S. growth 1950–2018: $y_0 = \$16{,}294$, $y_t = \$61{,}125$, $t = 68$

$$g = \left(\frac{61{,}125}{16{,}294}\right)^{1/68} - 1 \approx \boxed{1.94\% \text{ per year}}$$

Example 4 — Rule of 70 Application

Bank balance: $B_0 = \$100$

Time t	$r = 1\%$	$r = 6\%$
0	\$100	\$100
1	\$101	\$106
12	\$113	\$201
24	\$127	\$405
48	\$161	\$1,639
60	\$182	\$3,299

Rule of 70 check: at 6%, doubles every $70/6 \approx 12$ years. \$100 → \$200 at $t = 12$ ✓. After 60 years the **difference** is \$3,299 − \$182 = \$3,117 — from a mere 5 percentage points of interest rate difference.

Chapter 3 — Summary

Key Takeaways

1. Sustained growth in living standards is a very recent phenomenon — compressed into the last 2–3 centuries of human history.
2. The **Great Divergence**: before 1700 countries differed by at most 2×–3×; today by > 100×.
3. U.S. per capita GDP: \$3,600 (1870) → \$61,000 (2018); average growth ≈ 2%/yr.
4. **Constant growth rule:** $y_t = y_0(1+g)^t$
5. **Average growth rate:** $g = (y_t/y_0)^{1/t} - 1$
6. **Rule of 70:** Years to double $\approx 70/g$. Small differences in g compound to enormous differences over decades.
7. **Ratio scale:** constant growth = straight line; makes growth rates visually readable.
8. **Three growth rules:** (1) $g_{x/y} = g_x - g_y$; (2) $g_{xy} = g_x + g_y$; (3) $g_{x^a} = a\,g_x$
9. **Convergence:** Poor countries below their steady state grow faster; observed in OECD but not globally.
10. Growth rates 1960–2017 range from −2% to +7% across countries. Even small differences in g reshape living standards within one generation.

A Model of Production

4.1 Setup & Motivation

Central question: Per capita GDP varies by more than a factor of **50** between the richest and poorest countries. Why? This chapter builds a simple production model — the "ice cream economy" — and applies it to real cross-country data to explain income differences.

Definition

A **production function** describes how inputs (capital K, labor L) are combined to produce output Y:

$$Y = F(K, L)$$

Letters with a bar ($\bar{K}$, $\bar{L}$) denote **exogenous parameters** fixed by the model.

4.2 The Cobb-Douglas Production Function

The Key Equation

$$Y = F(K, L) = \bar{A}\, K^{1/3} L^{2/3} \tag{4.1}$$

Y	Total output (real GDP)
K	Capital stock (machines, buildings, equipment)
L	Labor (number of workers)
$\bar{A}$	**Total Factor Productivity (TFP)** — overall efficiency of production
$1/3$	Capital's exponent = capital's share of income
$2/3$	Labor's exponent = labor's share of income

4.2.1 Returns to Scale

Definition

Returns to scale: what happens to output when *all* inputs are scaled by $\lambda > 1$?

$$F(\lambda K, \lambda L) = \bar{A}(\lambda K)^{1/3}(\lambda L)^{2/3} = \bar{A}\,\lambda^{1/3+2/3}\,K^{1/3}L^{2/3} = \lambda\,Y$$

Since the exponents sum to $\frac{1}{3} + \frac{2}{3} = 1$, the Cobb-Douglas production function exhibits **constant returns to scale (CRS)**.

Returns to Scale	Condition	Example
Constant (CRS)	Exponents sum to $= 1$	$Y = K^{1/3}L^{2/3}$
Increasing (IRS)	Exponents sum to > 1	$Y = K^{2/3}L^{2/3}$
Decreasing (DRS)	Exponents sum to < 1	$Y = K^{1/3}L^{1/2}$

Why CRS? The "standard replication argument": if you double all inputs (build an identical factory, hire identical workers), you should be able to exactly double output. CRS is the natural baseline assumption for a whole economy.

4.2.2 Marginal Products

Key Equation

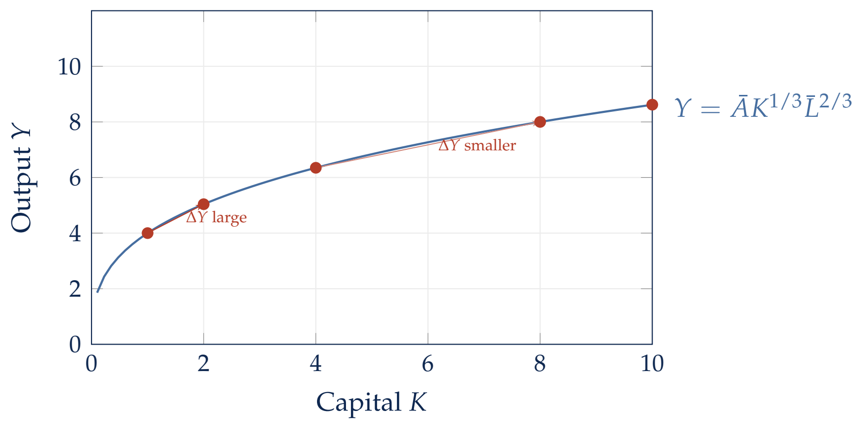
Marginal Product of Capital (MPK): extra output from one more unit of K, holding L fixed.

$$\text{MPK} = \frac{\partial Y}{\partial K} = \frac{1}{3} \cdot \bar{A} \cdot \left(\frac{L}{K}\right)^{2/3} = \frac{1}{3} \cdot \frac{Y}{K} \tag{4.3}$$

Marginal Product of Labor (MPL): extra output from one more unit of L, holding K fixed.

$$\text{MPL} = \frac{\partial Y}{\partial L} = \frac{2}{3} \cdot \bar{A} \cdot \left(\frac{K}{L}\right)^{1/3} = \frac{2}{3} \cdot \frac{Y}{L} \tag{4.4}$$

Diminishing marginal returns: MPK falls as K rises (with L fixed), because L/K decreases. MPL falls as L rises (with K fixed), because K/L decreases. Each additional unit of input adds less and less output.



Diminishing MPK: the slope of the production function flattens as K rises.

4.3 The Complete Production Model

4.3.1 Model Setup: 5 Equations, 5 Unknowns

Endogenous Variables (5): $Y,\ K,\ L,\ w,\ r$	
Production function	$Y = \bar{A}\,K^{1/3}L^{2/3}$
Capital market clearing	$K = \bar{K}$ (firms hire all available capital)
Labor market clearing	$L = \bar{L}$ (firms hire all available labor)
Firm's capital rule	$r = \text{MPK} = \frac{1}{3} \cdot \frac{Y}{K}$
Firm's labor rule	$w = \text{MPL} = \frac{2}{3} \cdot \frac{Y}{L}$
Exogenous Parameters: $\bar{A},\ \bar{K},\ \bar{L}$	

Firm's hiring rule: A profit-maximizing firm hires capital until MPK $= r$ and labor until MPL $= w$. If MPK $> r$, hiring another unit of capital adds more revenue than it costs, so the firm hires more. Equilibrium: MPK $= r$, MPL $= w$.

4.3.2 Equilibrium Solution

Table 4.2 — The Solution of the Production Model

Capital	$K^* = \bar{K}$
Labor	$L^* = \bar{L}$
Output	$Y^* = \bar{A}\,\bar{K}^{1/3}\bar{L}^{2/3}$
Rental rate	$r^* = \frac{1}{3} \cdot \frac{Y^*}{\bar{K}} = \frac{1}{3} \cdot \bar{A}\left(\frac{\bar{L}}{\bar{K}}\right)^{2/3}$
Wage	$w^* = \frac{2}{3} \cdot \frac{Y^*}{\bar{L}} = \frac{2}{3} \cdot \bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{1/3}$

4.3.3 Factor Shares of Income

From the equilibrium solution:

$$\frac{w^*L^*}{Y^*} = \frac{2}{3} \qquad \frac{r^*K^*}{Y^*} = \frac{1}{3} \tag{4.5}$$

Key Equation

$$w^*L^* + r^*K^* = Y^* \qquad \text{(National Income Identity)}$$

The 1/3–2/3 split is **not an assumption** — it is *derived* from profit maximization + Cobb-Douglas. It also matches the data: empirically, labor receives ≈ 2/3 and capital ≈ 1/3 of GDP in the U.S., which is why we chose the exponent 1/3 on capital.

4.4 Output Per Person: The Key Result

Divide $Y^* = \bar{A}\,\bar{K}^{1/3}\bar{L}^{2/3}$ by $\bar{L}$:

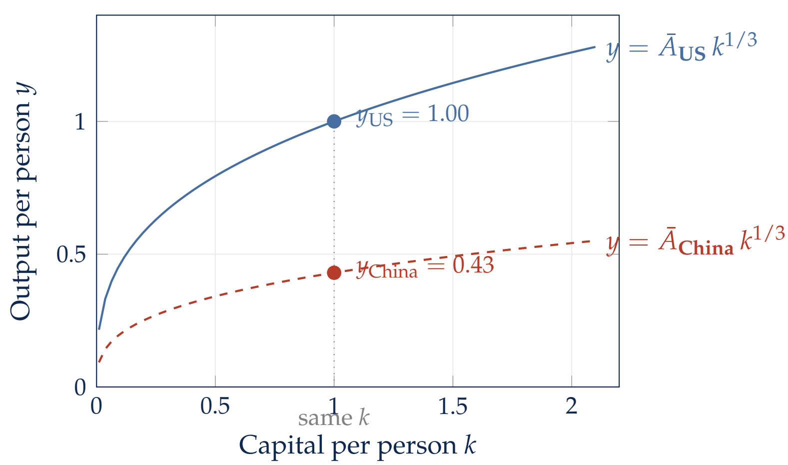
Per Capita Output in Equilibrium

$$y^* \equiv \frac{Y^*}{\bar{L}} = \bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{1/3} = \bar{A}\,k^{1/3} \tag{4.7}$$

where $y \equiv Y/L$ = output per person, $k \equiv K/L$ = capital per person.

$\bar{A}$ TFP	$\times$	$k^{1/3}$ Capital per person (diminishing returns)	$=$	y^* Per capita GDP

Capital market diagram:



At the *same* level of capital per person, a country with lower TFP produces substantially less output per person. The production function is shifted *down* by lower $\bar{A}$.

4.5 Development Accounting

Definition

Development accounting applies the production model to cross-country data to decompose income differences into contributions from (1) capital per person k and (2) total factor productivity $\bar{A}$.

4.5.1 Step 1: Assume $\bar{A} = 1$ everywhere

If all countries have the same TFP, then $y^* = k^{1/3}$.

Country	k (rel. U.S.)	$k^{1/3}$	**Actual** y	**Predicted** $y = k^{1/3}$
United States	1.000	1.000	1.000	1.000
Switzerland	1.094	1.030	1.151	1.030
Japan	0.976	0.992	0.734	0.992
China	0.651	0.867	0.279	0.867
India	0.433	0.757	0.117	0.757
Burundi	0.173	0.557	0.015	0.557

Problem: The model predicts Burundi's income should be $\approx 1/2$ of the U.S. In reality it is $\approx 1/64$. The model *vastly underpredicts* income differences when TFP is the same everywhere. Diminishing returns to capital are so strong that capital differences alone cannot explain the data.

4.5.2 Step 2: Back out implied TFP

Since $y^* = \bar{A}\,k^{1/3}$, we can solve for $\bar{A}$:

$$\bar{A} = \frac{y^*}{k^{1/3}}$$

Country	Per capita GDP (y)	$k^{1/3}$	Implied TFP ($\bar{A}$)
United States	1.000	1.000	1.000
Switzerland	1.151	1.094	**1.052**
U.K.	0.714	0.885	**0.807**
Japan	0.734	0.976	**0.752**
Italy	0.680	0.930	**0.731**
Spain	0.640	0.901	**0.710**
China	0.279	0.651	**0.429**
Brazil	0.252	0.587	**0.429**
South Africa	0.214	0.559	**0.383**
India	0.117	0.433	**0.270**
Burundi	0.015	0.173	**0.084**

4.5.3 Development Accounting Decomposition

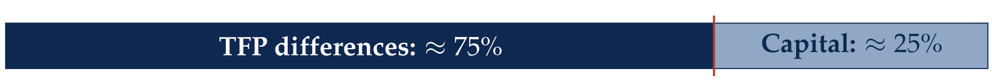
For any two countries, income differences decompose as:

Key Equation

$$\frac{y^*_{\text{rich}}}{y^*_{\text{poor}}} = \frac{\bar{A}_{\text{rich}}}{\bar{A}_{\text{poor}}} \times \left(\frac{k_{\text{rich}}}{k_{\text{poor}}}\right)^{1/3} \tag{4.10}$$

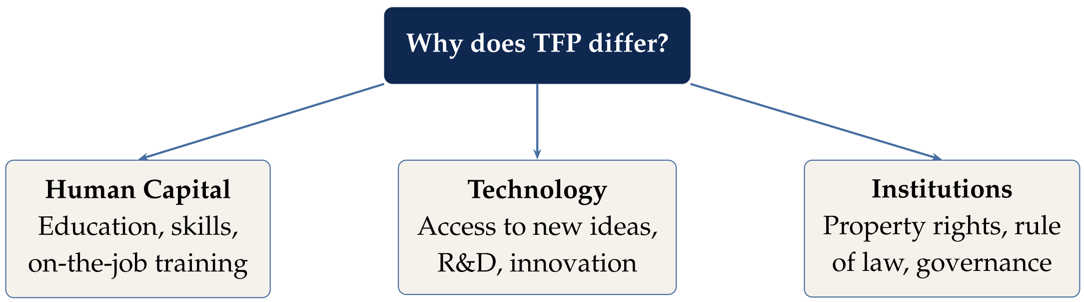
5 richest vs. 5 poorest countries (2017):

$$64 \approx \underbrace{13}_{\text{TFP}} \times \underbrace{5}_{\text{capital per person}} \tag{4.12}$$



Key finding: TFP explains $\approx$ **3/4** of income differences across countries; capital per person explains $\approx$ **1/4**. TFP is sometimes called the **residual** or **"a measure of our ignorance"** — we compute it by assuming the model holds, not by measuring it directly.

4.6 Why Does TFP Differ? Three Sources



Source	Key Points
Human Capital	Workers in rich countries have more education. In the U.S., each extra year of schooling raises wages $\approx 7\%$. Average schooling: $\sim$13 years (U.S.) vs. $\sim$4 years (poorest countries). Education differences explain *some* but not all of TFP gaps.
Technology	Poor countries often use older, less efficient technologies. Access to new ideas and innovation is unequal. Technology is embodied in both physical capital and human knowledge.
Institutions	Property rights, contract enforcement, and rule of law allow markets to function. Corruption, expropriation risk, and political instability reduce incentives to invest and innovate. Institutions are arguably the most fundamental explanation.

4.7 Worked Examples

Example 1 — Returns to Scale

Rule: Replace $(K, L) \to (\lambda K, \lambda L)$ and check if Y scales by λ, more, or less.

(a) $Y = K^{1/2}L^{1/2}$: $(2K)^{1/2}(2L)^{1/2} = 2^{1/2+1/2}K^{1/2}L^{1/2} = 2Y \quad \Rightarrow$ **CRS**

(b) $Y = K^{2/3}L^{2/3}$: $(2K)^{2/3}(2L)^{2/3} = 2^{4/3}Y > 2Y \quad \Rightarrow$ **IRS**

(c) $Y = K^{1/3}L^{1/2}$: Exponents: $1/3 + 1/2 = 5/6 < 1 \quad \Rightarrow$ **DRS**

(d) $Y = K^{1/3}L^{2/3} + A$: $2K^{1/3}L^{2/3} + A < 2(K^{1/3}L^{2/3} + A) \quad \Rightarrow$ **DRS**

Example 2 — Per Person Version of Production Function

Given $Y = \bar{A}K^{1/3}L^{2/3}$. Divide both sides by L:

$$\frac{Y}{L} = \bar{A}\frac{K^{1/3}L^{2/3}}{L} = \bar{A}\frac{K^{1/3}}{L^{1/3}} = \bar{A}\left(\frac{K}{L}\right)^{1/3}$$

$$\boxed{y = \bar{A}\,k^{1/3}}$$

where $y \equiv Y/L$ and $k \equiv K/L$. This is the per-person production function used throughout the chapter.

Example 3 — Development Accounting for Kenya

From data (2017, relative to U.S. = 1):

$$y_{\text{Kenya}} = 0.056, \qquad k_{\text{Kenya}}^{1/3} = 0.248$$

Implied TFP:

$$\bar{A}_{\text{Kenya}} = \frac{y}{k^{1/3}} = \frac{0.056}{0.248} \approx 0.226$$

U.S. is 17.9 times richer than Kenya. Decompose:

$$\frac{y_{\text{US}}}{y_{\text{Kenya}}} = \frac{\bar{A}_{\text{US}}}{\bar{A}_{\text{Kenya}}} \times \left(\frac{k_{\text{US}}}{k_{\text{Kenya}}}\right)^{1/3} = \frac{1}{0.226} \times \frac{1}{0.248} \approx 4.42 \times 4.03 \approx 17.8$$

Capital per person explains a factor of ≈ 4 (about $1/4$); TFP explains a factor of ≈ 4.4 (about $3/4$).

Example 4 — The Black Death (Wage Effect)

The Black Death (1347–1351) killed $\approx 1/3$ of Europe's population. What happened to wages?
Model prediction:

$$w^* = \frac{2}{3} \cdot \bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{1/3}$$

With L falling by $1/3$, the new labor force is $L' = \frac{2}{3}\bar{L}$:

$$\frac{w^*_{\text{new}}}{w^*_{\text{old}}} = \left(\frac{\bar{L}}{(2/3)\bar{L}}\right)^{1/3} = \left(\frac{3}{2}\right)^{1/3} \approx 1.145$$

Wages rose by approximately 14% — fewer workers each operated with more capital, raising MPL. This matches historical evidence of rising wages after the plague.

Chapter 4 — Summary

Key Takeaways

1. **Cobb-Douglas:** $Y = \bar{A}K^{1/3}L^{2/3}$. Output depends on TFP, capital, and labor.
2. **CRS:** Exponents sum to 1. Doubling all inputs doubles output.
3. **Diminishing returns:** MPK falls as K rises; MPL falls as L rises (holding the other input fixed).
4. **Firm's rule:** Hire until $\text{MPK} = r$ and $\text{MPL} = w$.
5. **Factor shares:** Labor gets $2/3$ of GDP; Capital gets $1/3$. This is both a model result and an empirical fact.
6. **Per person:** $y^* = \bar{A}\,k^{1/3}$. Per capita output depends on TFP and capital per person.
7. **Development accounting:** $\frac{y_{\text{rich}}}{y_{\text{poor}}} = \frac{A_{\text{rich}}}{A_{\text{poor}}} \times \left(\frac{k_{\text{rich}}}{k_{\text{poor}}}\right)^{1/3}$
8. **Result:** TFP explains $\approx 3/4$ of cross-country income differences; capital per person explains $\approx 1/4$.
9. **TFP sources:** Human capital, technology, and institutions — with institutions arguably the most fundamental.
10. TFP is also called the **residual** or "a measure of our ignorance" because we cannot measure it directly.

The Solow Growth Model

5.1 Overview & Motivation

What the Solow model adds to what we already know: The production model took capital $\bar{K}$ as *exogenous*. The Solow model **endogenizes** capital accumulation — it explains where the capital stock comes from and how it evolves over time.
Central insight: Diminishing returns to capital mean that capital accumulation alone *cannot* sustain long-run growth. But the model *can* explain differences in income levels and growth rates across countries via the **principle of transition dynamics**.

5.2 Setting Up the Model: 5 Equations

The Five Equations of the Solow Model

$$Y_t = \bar{A}K_t^{1/3}L^{2/3} \quad (5.1)$$

Production function

$$C_t + I_t = Y_t \quad (5.2)$$

Resource constraint

$$\Delta K_{t+1} = I_t - \bar{d}K_t \quad (5.3)$$

Capital accumulation

$$I_t = \bar{s}\,Y_t \quad (5.4)$$

Investment allocation

$$L_t = \bar{L} \quad \text{(fixed)}$$

Labor (exogenous)

Symbol	Meaning	**Type**	Notes
Y_t	Output	**Endogenous**	From production function
K_t	Capital stock	**Endogenous**	Accumulates over time
C_t	Consumption	**Endogenous**	$(1-\bar{s})Y_t$
I_t	Investment	**Endogenous**	$\bar{s}Y_t$
$\bar{A}$	TFP	**Parameter**	Exogenous efficiency
$\bar{s}$	Investment rate	**Parameter**	$0 < \bar{s} < 1$
$\bar{d}$	Depreciation rate	**Parameter**	Typically 0.07–0.10
$\bar{L}$	Labor	**Parameter**	Fixed, exogenous

Capital accumulation equation (combining 5.3 and 5.4):

$$\Delta K_{t+1} = \bar{s}\, Y_t - \bar{d}\, K_t$$

The capital stock *rises* when investment $\bar{s}Y_t$ exceeds depreciation $\bar{d}K_t$, and *falls* when depreciation exceeds investment. Net investment $= \bar{s}Y_t - \bar{d}K_t$.

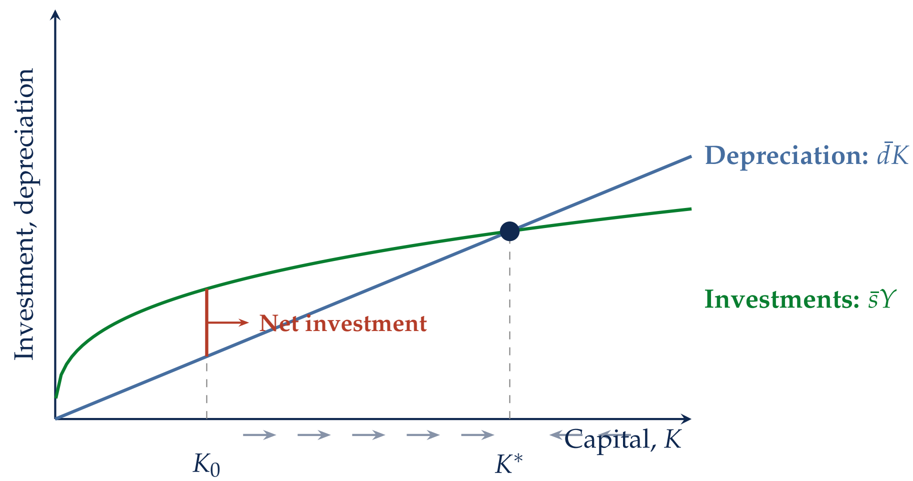
5.2.1 Capital Accumulation Example

t	K_t	I_t	$\bar{d}K_t$	ΔK_{t+1}
0	1,000	200	100	**100**
1	1,100	200	110	**90**
2	1,190	200	119	**81**
3	1,271	200	127	**73**
4	1,344	200	134	**66**
5	1,410	200	141	**59**

ΔK_{t+1} shrinks each period because depreciation rises with capital while investment ($I = 200$) is fixed. This is the force that drives the economy toward a steady state.

5.3 The Solow Diagram

The Solow diagram plots the two forces governing ΔK_{t+1}: investment $\bar{s}Y$ (curved, concave) and depreciation $\bar{d}K$ (straight line).

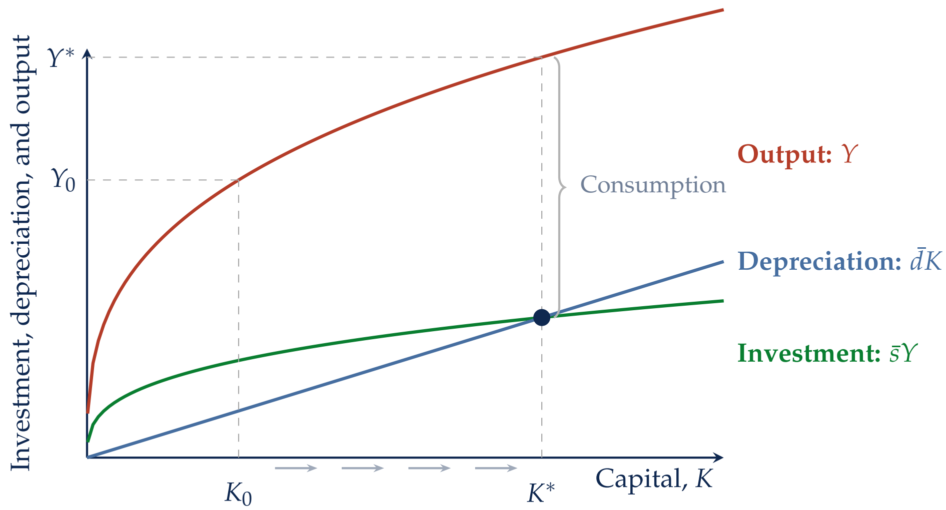


The Solow diagram: investment $\bar{s}Y$ (concave curve) and depreciation $\bar{d}K$ (straight line). At K^*, the two curves intersect: $\bar{s}Y^* = \bar{d}K^*$ — the steady state. Arrows show transition dynamics.

Region	What happens
$K_0 < K^*$	$\bar{s}Y > \bar{d}K \Rightarrow$ net investment $> 0 \Rightarrow K$ rises $\to K^*$
$K_0 = K^*$	$\bar{s}Y = \bar{d}K \Rightarrow$ net investment $= 0 \Rightarrow K$ stays constant
$K_0 > K^*$	$\bar{s}Y < \bar{d}K \Rightarrow$ net investment $< 0 \Rightarrow K$ falls $\to K^*$

5.4 The Solow Diagram with Output

Adding the production function $Y = \bar{A}K^{1/3}\bar{L}^{2/3}$ to the diagram:



The Solow diagram with output. As K rises from K_0 to K^*, output rises from Y_0 to Y^*. Consumption is the vertical gap between output Y and investment $\bar{s}Y$.

5.5 Solving for the Steady State

Definition

The **steady state** is the long-run rest point of the Solow model, where $\Delta K_{t+1} = 0$:

$$\bar{s}\,Y^* = \bar{d}\,K^*$$

Investment exactly offsets depreciation. The capital stock and output remain constant.

5.5.1 Mathematical Solution

Step 1: Set $\bar{s}Y^* = \bar{d}K^*$ and substitute the production function:

$$\bar{s}\,\bar{A}\,(K^*)^{1/3}\bar{L}^{2/3} = \bar{d}\,K^*$$

Step 2: Collect K^* terms and raise both sides to the $3/2$ power:

Steady-State Solutions

$$K^* = \left(\frac{\bar{s}\,\bar{A}}{\bar{d}}\right)^{3/2} \bar{L} \tag{5.7}$$

$$Y^* = \left(\frac{\bar{s}}{\bar{d}}\right)^{1/2} \bar{A}^{3/2}\bar{L} \tag{5.8}$$

$$y^* \equiv \frac{Y^*}{\bar{L}} = \bar{A}^{3/2}\left(\frac{\bar{s}}{\bar{d}}\right)^{1/2} \tag{5.9}$$

Comparative statics — how parameters affect y^*:

Parameter	Effect on y^*	Intuition
$\bar{s}\uparrow$	$\uparrow$	More saving $\to$ more investment $\to$ more capital $\to$ higher output
$\bar{A}\uparrow$	$\uparrow$ (exponent 3/2)	Higher TFP raises output *directly* and *indirectly* via more capital
$\bar{d}\uparrow$	$\downarrow$	Faster depreciation erodes capital stock
$\bar{L}\uparrow$	No effect on y^*	More people $\to$ more Y^*, but $y^* = Y^*/L$ is unchanged

Key difference from Chapter 4: In the production model, $y^* = \bar{A}\,k^{1/3}$. In the Solow model, $y^* = \bar{A}^{3/2}(\bar{s}/\bar{d})^{1/2}$. The exponent on $\bar{A}$ is $3/2 > 1$ in the Solow model, because higher TFP also induces more capital accumulation. TFP is *even more important* in the Solow model.

5.5.2 Capital-Output Ratio at Steady State

From $\bar{s}Y^* = \bar{d}K^*$:

Key Equation

$$\frac{K^*}{Y^*} = \frac{\bar{s}}{\bar{d}} \tag{5.11}$$

Countries with higher investment rates $\bar{s}$ should have higher capital-output ratios. **This prediction holds strongly in the data.**

5.6 No Long-Run Growth in the Solow Model

The central failure of the Solow model: In the long run, the economy converges to a constant K^* and Y^*. There is **no sustained growth** in per capita output.
Why? Diminishing returns to capital. Each extra unit of K adds less and less output, so investment grows slower than depreciation eventually catches up. At K^*: $\bar{s}Y = \bar{d}K$ and $\Delta K = 0$.
Implication: Capital accumulation is *not* the engine of long-run growth. Something else is needed $\rightarrow$ Chapter 6 (ideas and TFP growth).

5.7 Principle of Transition Dynamics

Definition

Principle of Transition Dynamics: The farther an economy is *below* its steady state, the **faster** it will grow. The farther it is *above* its steady state, the faster it will *shrink* back.

Economy **below** K^*
$\bar{s}Y > \bar{d}K$
$\Delta K > 0 \rightarrow$ **growth**

$\rightarrow$

Economy **at** K^*
$\bar{s}Y = \bar{d}K$
$\Delta K = 0 \rightarrow$ **steady**

$\leftarrow$

Economy **above** K^*
$\bar{s}Y < \bar{d}K$
$\Delta K < 0 \rightarrow$ **shrinks**

Empirical application — South Korea vs. Philippines:

	South Korea	**Philippines**
Investment rate $\bar{s}$	Rose from ∼10% to >35% (1960–1990)	Remained stable ∼20%
Implication	Steady state K^* shifted up sharply	Steady state unchanged
Prediction	Rapid growth (transition dynamics)	Slow growth
Actual outcome	≈ 6%/yr growth, 1950–1990	Slow growth

5.8 Economic Experiments: Shifting the Curves

Shock	Which curve shifts?	Effect on K^*, Y^*
$\bar{s}\uparrow$ **(investment rate rises)**	$\bar{s}Y$ curve shifts **up**	$K^*\uparrow$, $Y^*\uparrow$
$\bar{d}\uparrow$ **(depreciation rises)**	$\bar{d}K$ line rotates **up**	$K^*\downarrow$, $Y^*\downarrow$
$\bar{A}\uparrow$ **(TFP rises)**	$\bar{s}Y$ curve shifts **up**	$K^*\uparrow$, $Y^*\uparrow$
$\bar{L}\uparrow$ **(population rises)**	$\bar{s}Y$ curve shifts **up**	$K^*\uparrow$, $Y^*\uparrow$, but y^* unchanged
Earthquake (*K* drops)	No shift — movement **along** curves	Economy returns to same K^*

Key distinction: A change in a *parameter* ($\bar{s}$, $\bar{d}$, $\bar{A}$) shifts a curve and changes the steady state. A change in the *capital stock* K (e.g., earthquake) is a movement *along* the curves with no change in K^*. The economy will simply transition back to the original K^*.

5.9 Development Accounting in the Solow Model

Using $y^* = \bar{A}^{3/2}(\bar{s}/\bar{d})^{1/2}$, income differences decompose as:

Key Equation

$$\frac{y^*_{\text{rich}}}{y^*_{\text{poor}}} = \left(\frac{\bar{A}_{\text{rich}}}{\bar{A}_{\text{poor}}}\right)^{3/2} \times \left(\frac{\bar{s}_{\text{rich}}}{\bar{s}_{\text{poor}}}\right)^{1/2}$$

5 richest vs. 5 poorest countries:

Investment rates: rich $\approx 28\%$, poor $\approx 7\%$ $\Rightarrow$ ratio $= 4 \Rightarrow$ investment differences explain $\sqrt{4} = 2\times$ of income gap.

Total income gap $= 64\times \Rightarrow$ TFP explains $64/2 = 32\times$.

$$64 \approx \underbrace{32}_{\text{TFP (exponent 3/2)}} \times \underbrace{2}_{\text{investment rate}}$$

In the Solow model, TFP is **even more important** than in Chapter 4's production model (where TFP explained a factor of 13 vs. 32 here). This is because TFP now also affects the capital stock — higher $\bar{A}$ induces more capital accumulation, amplifying its effect on income.

5.10 Worked Examples

Example 1 — Solving for Steady State

Given: $\bar{A} = 1$, $\bar{s} = 0.2$, $\bar{d} = 0.1$, $\bar{L} = 100$
Steady-state capital:

$$K^* = \left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{3/2} \bar{L} = \left(\frac{0.2}{0.1}\right)^{3/2} \times 100 = 2^{3/2} \times 100 = 2\sqrt{2} \times 100 \approx 283$$

Steady-state output per person:

$$y^* = \bar{A}^{3/2} \left(\frac{\bar{s}}{\bar{d}}\right)^{1/2} = 1^{3/2} \times \left(\frac{0.2}{0.1}\right)^{1/2} = \sqrt{2} \approx \boxed{1.414}$$

Capital-output ratio:

$$\frac{K^*}{Y^*} = \frac{\bar{s}}{\bar{d}} = \frac{0.2}{0.1} = \boxed{2}$$

Example 2 — Earthquake (Transition Dynamics)

Economy at steady state K^*. Earthquake destroys half the capital stock: $K_{\text{new}} = K^*/2$.
Effect on output immediately:

$$Y_{\text{new}} = \bar{A} \left(\frac{K^*}{2}\right)^{1/3} \bar{L}^{2/3} = \left(\frac{1}{2}\right)^{1/3} Y^* \approx 0.794\, Y^*$$

Output falls by only $\approx 20\%$ even though capital fell by 50% — because of the $1/3$ exponent (diminishing returns).
Afterward: At $K^*/2 < K^*$, investment exceeds depreciation $\Rightarrow$ economy grows back toward the *original* K^* (no shift in curves, no change in steady state).

Example 3 — Increase in Investment Rate

Economy starts at K^* with $\bar{s} = 0.2$. Policy raises $\bar{s}$ to 0.3 permanently.
New steady state:

$$\frac{y^{**}}{y^*} = \left(\frac{\bar{s}_{\text{new}}}{\bar{s}_{\text{old}}}\right)^{1/2} = \left(\frac{0.3}{0.2}\right)^{1/2} = \sqrt{1.5} \approx \boxed{1.22}$$

Per capita output rises by 22% in the long run. Growth is fastest immediately after the change, then slows as the economy converges to y^{**}.

5.11 Strengths and Weaknesses of the Solow Model

Strengths	Weaknesses
Determines long-run income level via $y^* = \bar{A}^{3/2}(\bar{s}/\bar{d})^{1/2}$	**Cannot explain sustained long-run growth**
Capital-output ratio $= \bar{s}/\bar{d}$ fits data well	**Investment rates explain only a small fraction of income differences**
Principle of transition dynamics explains cross-country growth rate differences	**TFP differences remain unexplained (the "residual")**
Convergence within OECD explained by transition dynamics	**No theory of why TFP differs across countries**

Chapter 5 — Summary

Key Takeaways

1. Solow adds capital accumulation to Chapter 4: $\Delta K_{t+1} = \bar{s}Y_t - \bar{d}K_t$.
2. **Solow diagram:** $\bar{s}Y$ (concave) vs. $\bar{d}K$ (linear). Intersection = steady state K^*.
3. **Steady state:** $\bar{s}Y^* = \bar{d}K^* \Rightarrow y^* = \bar{A}^{3/2}(\bar{s}/\bar{d})^{1/2}$.
4. **Capital-output ratio:** $K^*/Y^* = \bar{s}/\bar{d}$ — confirmed strongly in cross-country data.
5. **No long-run growth:** Diminishing returns to capital $\Rightarrow$ economy converges to a constant y^*. Capital accumulation is not the engine of long-run growth.
6. **Principle of transition dynamics:** Farther below $K^* \Rightarrow$ faster growth. Explains why countries like South Korea and Japan grew rapidly (rising $\bar{s}$, moving toward higher K^*).
7. **Curve shifts:** $\bar{s} \uparrow$ or $\bar{A} \uparrow$ shifts $\bar{s}Y$ up (new higher K^*); $\bar{d} \uparrow$ rotates $\bar{d}K$ up (new lower K^*). Shocks to K only move along curves.
8. TFP exponent is $3/2$ in Solow (vs. 1 in Ch. 4) because higher $\bar{A}$ also induces more capital. TFP is even more important here.
9. Income gap decomposition: $64 \approx 32 \times 2$ (TFP $\times$ investment rates).

Growth and Ideas

6.1 The Big Idea: Objects vs. Ideas

Why the Solow model fails at long-run growth: Capital is an *object* — it runs into diminishing returns, so capital accumulation eventually stops. Romer's insight: **ideas** are fundamentally different from objects. The nonrivalry of ideas allows for sustained growth.

	Objects	**Ideas**
Examples	Capital, labor, land, oil, computers	Recipes, designs, formulas, algorithms, management techniques
Rival?	**Yes** — one person's use excludes another's	**No** — can be used simultaneously by everyone
Returns	Constant returns to scale	Nonrivalry $\Rightarrow$ increasing returns overall
Scarce?	Yes — limited by supply	Not in the same way — a used idea is not depleted

Definition

Nonrivalry: A good is nonrival if one person's use does not diminish another's ability to use it simultaneously.
Standard replication argument: Doubling all *objects* (capital, labor) should double output $\Rightarrow$ constant returns to objects alone. But since ideas are nonrival and can be used in both copies, there are **increasing returns** to objects and ideas *together*.

6.2 Increasing Returns and the Problem with Competition

The key tension: Ideas have large *fixed costs* (R&D) but near-zero marginal cost of replication. If firms charge price = marginal cost (as perfect competition requires), they can never recoup the fixed cost of research $\Rightarrow$ no innovation.
Implication: Markets with increasing returns require *imperfect competition* — firms must charge prices above marginal cost to fund innovation. Adam Smith's invisible hand may not deliver the socially optimal amount of research.

Nonrivalry of ideas → **Increasing** returns to scale → **Fixed costs** of innovation → **Imperfect** competition needed

6.3 The Romer Model

6.3.1 Setup: 4 Equations, 4 Unknowns

The Four Equations of the Romer Model

Output production function

$$Y_t = A_t L_{yt} \tag{6.2}$$

Ideas production function

$$\Delta A_{t+1} = \bar{z} A_t L_{at} \tag{6.3}$$

Labor resource constraint

$$L_{yt} + L_{at} = \bar{L} \tag{6.4}$$

Research allocation

$$L_{at} = \bar{\ell}\,\bar{L} \tag{6.5}$$

Symbol	Meaning	**Type**	Notes
Y_t	Output (consumption good)	**Endogenous**	
A_t	Stock of ideas (knowledge)	**Endogenous**	Grows over time
L_{yt}	Workers in goods sector	**Endogenous**	$(1-\bar{\ell})\bar{L}$
L_{at}	Researchers	**Endogenous**	$\bar{\ell}\bar{L}$
$\bar{z}$	Research productivity	**Parameter**	
$\bar{\ell}$	Research share of labor	**Parameter**	$0 < \bar{\ell} < 1$
$\bar{L}$	Total labor	**Parameter**	Fixed

6.3.2 Key Features of the Two Production Functions

	Output: $Y_t = A_t L_{yt}$	**Ideas:** $\Delta A_{t+1} = \bar{z} A_t L_{at}$
Returns to objects (L)	CRS in L alone (exponent $= 1$)	CRS in L alone
Returns to ideas (A)	Increasing (exponent on $A = 1$)	No diminishing returns
Returns to both	**Increasing returns overall**	**Increasing returns overall**
Key property	Nonrivalry of A_t	No diminishing returns to ideas

Why no diminishing returns to ideas? In the ideas production function, the exponent on A_t is exactly 1. Old ideas help produce new ideas in a *virtuous circle* — the more we know, the easier it is to discover more. This is fundamentally different from capital in the Solow model, where diminishing returns eventually kill growth.

6.3.3 Solving the Romer Model

Step 1 — Growth rate of knowledge:

From the ideas equation with $L_{at} = \bar{\ell}\bar{L}$:

$$\frac{\Delta A_{t+1}}{A_t} = \bar{z}\, L_{at} = \bar{z}\, \bar{\ell}\, \bar{L}$$

Growth Rate of Knowledge (Balanced Growth Path)

$$g \equiv \frac{\Delta A_{t+1}}{A_t} = \bar{z}\, \bar{\ell}\, \bar{L} \tag{6.7}$$

The growth rate of knowledge is **constant** — determined by research productivity $\bar{z}$, research share $\bar{\ell}$, and population size $\bar{L}$.

Step 2 — Level of knowledge over time:

$$A_t = A_0(1+g)^t \tag{6.8}$$

Step 3 — Output per person:

$L_{yt} = (1-\bar{\ell})\bar{L}$, so $Y_t = A_t(1-\bar{\ell})\bar{L}$. Divide by $\bar{L}$:

Output Per Person in the Romer Model

$$y_t \equiv \frac{Y_t}{\bar{L}} = A_0\,(1-\bar{\ell})\,(1+g)^t \tag{6.9}$$

where $g = \bar{z}\, \bar{\ell}\, \bar{L}$.

Key result: Per capita output grows at rate $g = \bar{z}\bar{\ell}\bar{L}$ *forever* — there is **sustained growth**. This is what the Solow model could not deliver. The growth rate never falls because there are no diminishing returns to ideas.

6.3.4 Balanced Growth Path

Definition

A **balanced growth path** is a situation where all endogenous variables grow at constant rates. Unlike the Solow model, the Romer model has no transition dynamics — the economy is on its balanced growth path from the start, always growing at rate g.

6.4 Two Experiments in the Romer Model

6.4.1 Experiment 1: Increase in Population $\bar{L}$

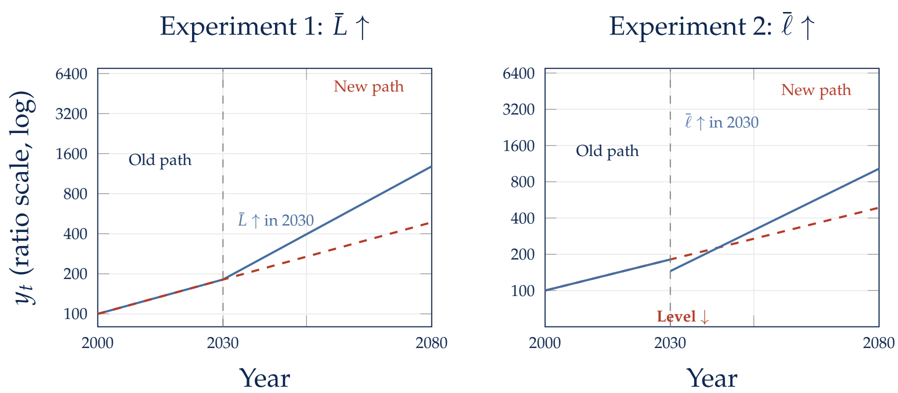
Effect on growth rate	$g = \bar{z}\bar{\ell}\bar{L}$ rises immediately and permanently
Effect on level	Output per person level also rises (jumps up)
Intuition	More people $\Rightarrow$ more researchers $\Rightarrow$ more ideas $\Rightarrow$ faster growth
On ratio scale	Straight line tilts upward (steeper slope) permanently

This is a **growth effect** — a permanent change in the *growth rate*. The Romer model predicts that larger economies (more $\bar{L}$) should permanently grow faster. This is a strong prediction that may not fully match data — an extension where the idea exponent is less than 1 produces only a **level effect** instead.

6.4.2 Experiment 2: Increase in Research Share $\bar{\ell}$

Effect on growth rate	$g = \bar{z}\bar{\ell}\bar{L}$ rises permanently
Effect on level (short run)	y_t *falls* initially: fewer workers produce goods
Effect on level (long run)	Higher growth $\Rightarrow$ higher future y_t
Trade-off	Current consumption $\downarrow$ vs. future consumption $\uparrow$
On ratio scale	Level drops then grows faster (higher slope)

Illustrative ratio-scale diagrams:



6.4.3 Growth Effects vs. Level Effects

	Growth Effect	**Level Effect**
Definition	Permanent change in the *growth rate*	Permanent change in the *level* only
Arises when	Idea exponent $=$ 1 (standard Romer)	Idea exponent $<$ 1 (diminishing returns to ideas)
Example	$\bar{L}\uparrow$ or $\bar{\ell}\uparrow$ in basic Romer	Same shock in extended Romer
Which is stronger?	Much larger long-run effect	Smaller but still permanent effect

6.5 Growth Accounting

Definition

Growth accounting decomposes the growth rate of output into contributions from capital, labor, and TFP, using the production function and growth rate rules.

Starting from the production function:

$$Y_t = A_t K_t^{1/3} L_{yt}^{2/3} \tag{6.10}$$

Apply growth rate rules (Chapter 3):

Growth Accounting Equation

$$\underbrace{g_Y}_{\text{output growth}} = \underbrace{g_A}_{\text{TFP growth}} + \underbrace{\tfrac{1}{3}g_K}_{\text{capital contrib.}} + \underbrace{\tfrac{2}{3}g_L}_{\text{labor contrib.}} \tag{6.11}$$

Subtracting g_L from both sides (to get per-worker growth):

$$\underbrace{g_Y - g_L}_{\text{growth of } Y/L} = \underbrace{\tfrac{1}{3}(g_K - g_L)}_{\text{contrib. of } K/L} + \underbrace{\tfrac{2}{3}(g_{L_y} - g_L)}_{\text{labor composition}} + \underbrace{g_A}_{\text{TFP growth}} \tag{6.12}$$

TFP as the residual: In practice, we can observe Y, K, and L. Everything except g_A is measurable, so we compute:

$$g_A = g_Y - \frac{1}{3}g_K - \frac{2}{3}g_L$$

TFP growth is computed as a residual — it captures everything *not* explained by observable inputs. For this reason, it is also called "the Solow residual."

U.S. Growth Accounting (Table 6.2):

(avg. annual %, all figures)	**1948–2017**	**1948–73**	**1973–95**	**1995–03**	**2003–17**
Output per hour (Y/L)	2.3	3.3	1.5	3.2	1.4
Contrib. of K/L	0.9	1.0	0.8	1.4	0.5
Labor composition	0.2	0.2	0.2	0.3	0.2
TFP growth (g_A)	**1.2**	**2.1**	**0.5**	**1.5**	**0.7**

Key episodes:

- **Productivity slowdown (1973–95):** TFP collapsed from 2.1% to 0.5%. Causes debated: oil shocks, declining R&D share, sectoral shift toward services.
- **New economy boom (1995–2003):** TFP rebounded to 1.5%. Driven by IT revolution and internet. Computers explain at least half.
- **Post-2003 slowdown:** TFP fell to 0.7%. Possibly a legacy of the Great Recession or secular stagnation.

TFP accounts for the majority of growth in all periods.

6.6 Combining Solow and Romer

Definition

The combined Solow–Romer model adds capital accumulation back into the Romer model, giving a richer and more complete theory of long-run growth.

The Combined Model (5 Equations, 5 Unknowns)

Output production function

$$Y_t = A_t K_t^{1/3} L_{yt}^{2/3} \tag{6.13}$$

Capital accumulation

$$\Delta K_{t+1} = \bar{s} Y_t - \bar{d} K_t \tag{6.14}$$

Ideas production function

$$\Delta A_{t+1} = \bar{z} A_t L_{at} \tag{6.15}$$

Labor resource constraint

$$L_{yt} + L_{at} = \bar{L} \tag{6.16}$$

Research allocation

$$L_{at} = \bar{\ell}\,\bar{L} \tag{6.17}$$

Unknowns: $Y_t,\ K_t,\ A_t,\ L_{yt},\ L_{at}$

Long-run balanced growth path:

Since A_t grows at rate $g = \bar{z}\bar{\ell}\bar{L}$ continuously, the $\bar{s}Y$ curve in the Solow diagram shifts up every period. Capital never settles to a fixed steady state but instead grows at rate g as well. On the balanced growth path, capital, output, and knowledge all grow at the same constant rate g.

Variable	Long-run growth rate
Stock of ideas A_t	$g = \bar{z}\bar{\ell}\bar{L}$
Capital K_t	g
Output Y_t	g
Output per person y_t	g

Role of transition dynamics in combined model: Countries can grow faster or slower than the world rate g because of the principle of transition dynamics (from Solow). Countries below their balanced growth path grow faster; those above grow slower.
Summary: The Romer model explains the *trend* (why $g > 0$ permanently). The Solow model explains deviations from the trend (why some countries grow faster or slower at any given time).

6.7 Worked Examples

Example 1 — Calculating Growth Rate and Output Level

Given: $\bar{z} = 0.10$, $\bar{\ell} = 1/500$, $\bar{L} = 100$, $A_0 = 100$
Growth rate of knowledge:

$$g = \bar{z}\bar{\ell}\bar{L} = 0.10 \times \frac{1}{500} \times 100 = \boxed{0.02 = 2\% \text{ per year}}$$

Output per person at $t = 0$:

$$y_0 = A_0(1 - \bar{\ell}) = 100 \times (1 - 0.002) \approx \boxed{99.8}$$

Output per person at $t = 100$:

$$y_{100} = 100 \times (1 - 0.002) \times (1.02)^{100} \approx 99.8 \times 7.245 \approx \boxed{723}$$

Example 2 — Doubling the Research Share

Using same parameters but now $\bar{\ell}' = 2/500 = 0.004$.
New growth rate:

$$g' = 0.10 \times 0.004 \times 100 = \boxed{4\% \text{ per year}}$$

New output at $t = 0$ (fewer goods workers):

$$y_0' = 100 \times (1 - 0.004) = \boxed{99.6} \quad \text{(slightly lower)}$$

New output at $t = 100$:

$$y_{100}' = 100 \times (1 - 0.004) \times (1.04)^{100} \approx 99.6 \times 50.5 \approx \boxed{5{,}030}$$

Comparison: $y_{100}' \approx 5{,}030$ vs. $y_{100} \approx 723$. Doubling research is enormously valuable in the long run, despite a small short-run cost.

Example 3 — Growth Accounting for the U.S.

Given (1948–2017): $g_Y = 3.5\%$, $g_K = 3.8\%$, $g_L = 1.7\%$
TFP growth (residual):

$$g_A = g_Y - \frac{1}{3}g_K - \frac{2}{3}g_L = 3.5 - \frac{1}{3}(3.8) - \frac{2}{3}(1.7) = 3.5 - 1.27 - 1.13 = \boxed{1.1\%}$$

Interpretation: About half of all output growth is explained by TFP (ideas and efficiency), not by measurable capital or labor inputs.

Chapter 6 — Summary

Key Takeaways

1. **Objects vs. ideas:** Objects are rival (scarce); ideas are nonrival (can be used by everyone simultaneously without depletion).
2. **Nonrivalry $\Rightarrow$ increasing returns:** Standard replication argument gives CRS to objects; nonrival ideas add on top $\Rightarrow$ IRS to objects and ideas together.
3. **IRS $\Rightarrow$ imperfect competition:** Fixed costs of R&D cannot be covered at marginal cost pricing. Markets may provide too little innovation.
4. **Romer model:** $Y_t = A_t L_{yt}$; $\Delta A_{t+1} = \bar{z} A_t L_{at}$; $L_{at} = \bar{\ell}\bar{L}$.
5. **Sustained growth:** $g = \bar{z}\bar{\ell}\bar{L}$; $y_t = A_0(1-\bar{\ell})(1+g)^t$. Growth never stops because there are no diminishing returns to ideas.
6. **Experiment 1 ($\bar{L}\uparrow$):** Growth rate rises permanently (growth effect). Level of y also rises.
7. **Experiment 2 ($\bar{\ell}\uparrow$):** Growth rate rises; current level falls (fewer goods workers). Long-run gains are large.
8. **No transition dynamics** in Romer: the economy is always on its balanced growth path.
9. **Growth accounting:** $g_Y = g_A + \frac{1}{3}g_K + \frac{2}{3}g_L$. TFP is the residual and accounts for the majority of growth.
10. **Combined Solow–Romer:** Romer explains the growth trend ($g = \bar{z}\bar{\ell}\bar{L}$). Solow's transition dynamics explain deviations from the trend across countries.