Calculus I (MAT 135)

Lecture Notes

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Contents

Chapter 1: Limits and Continuity

1.1 Introduction to Limits

Big idea!
A limit describes the behavior of $f(x)$ *near* a point $x = c$. It does not require that $f(c)$ exists, and it does not require that $f(c)$ equals the limit.

In calculus, we are often interested not only in the value of a function at a point, but in how the function behaves near that point. This idea leads to the concept of a *limit*.

1.1.1 What Is a Limit?

Let $f(x)$ be a function. The expression

$$\lim_{x \to c} f(x) = L$$

means that as x gets closer and closer to c, the values of $f(x)$ get closer and closer to L. It is important to note that the limit depends on the behavior of the function near $x = c$, not necessarily on the value of the function at $x = c$.

1.1.2 Approximating Limits Using Tables

One way to understand limits is by using a table of values. We choose values of x close to c, from both the left and the right, and observe the behavior of $f(x)$.

Example 1

Approximate the limit

$$\lim_{x \to 2} (x^2 - 4).$$

We construct a table of values near $x = 2$:

x	$x^2 - 4$
1.9	−0.39
1.99	−0.0399
1.999	−0.003999
2.001	0.004001
2.01	0.0401
2.1	0.41

As x approaches 2 from both sides, the values of $x^2 - 4$ approach 0. Therefore,

$$\lim_{x \to 2} (x^2 - 4) = 0.$$

Example 2

Approximate the limit

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

x	$\frac{\sin x}{x}$
−0.1	0.998334
−0.01	0.999983
−0.001	0.999999
0.001	0.999999
0.01	0.999983
0.1	0.998334

Even though $\frac{\sin x}{x}$ is undefined at $x = 0$, the values of the function approach 1 as x approaches 0. Thus,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

1.1.3 Approximating Limits Using Graphs

Graphs provide a visual way to understand limits by showing how the function behaves near a point.

Example 1

Consider the graph of a function with an open circle at $(0, 1)$ and a smooth curve approaching that point from both sides. Although the function may not be defined at $x = 0$, the graph shows that the y-values approach 1 as x approaches 0. Therefore,

$$\lim_{x \to 0} f(x) = 1.$$

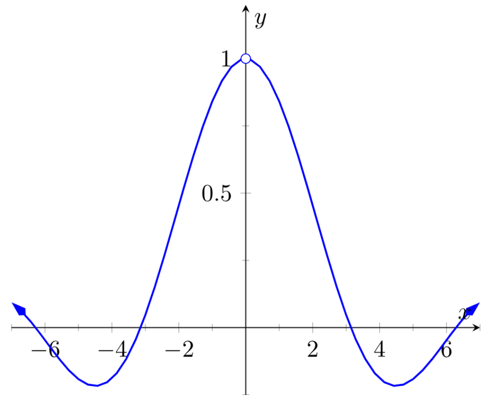


Figure 1.1: The graph of a function approaching a value as x approaches 0. The open circle indicates that the function is not defined at that point, even though the limit exists.

Example 2

Consider a graph where the function forms a smooth peak at $y = 1$ near $x = 0$, but the point $(0, 1)$ is not filled in. From both the left and right, the function values approach 1. Hence, the limit exists and equals 1, even if $f(0)$ is undefined.

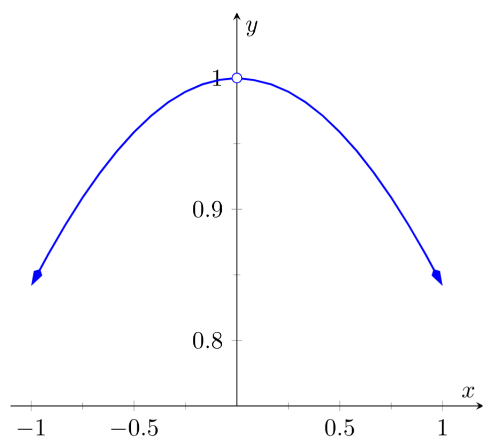


Figure 1.2: A wider view of the same function. Although the graph varies significantly away from $x = 0$, the values of the function near $x = 0$ still approach 1, illustrating that limits depend only on local behavior.

1.1.4 Approximating Limits Algebraically

For many basic functions, limits can be found by direct substitution, provided the function is continuous at the point.

Examples 1 & 2

$$\lim_{x\to 3}(x^2+1) = 3^2+1 = 10.$$

$$\lim_{x\to -1}(5x-2) = 5(-1)-2 = -7.$$

Approximating Limits Using Tables and Graphs – Extra Examples

Example 1: Approximating the Value of a Limit

Use graphical and numerical methods to approximate

$$\lim_{x\to 3}\frac{x^2-x-6}{6x^2-19x+3}.$$

Graphical Approximation. We graph the function

$$y = \frac{x^2-x-6}{6x^2-19x+3}$$

on a small interval containing $x = 3$. The graph shows that as x approaches 3 from both sides, the values of the function approach approximately 0.29, even though the function is not defined at $x = 3$.

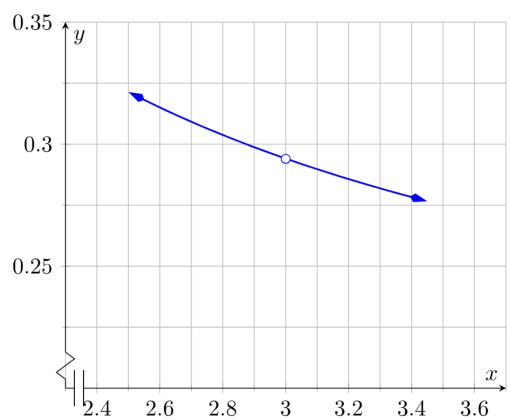


Figure 1.3: Graph of $y = \dfrac{x^2-x-6}{6x^2-19x+3}$ near $x = 3$.

Numerical Approximation. We now construct a table of values for x near 3.

x	$\frac{x^2 - x - 6}{6x^2 - 19x + 3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	not defined
3.001	0.294073
3.01	0.293669
3.1	0.289773

From both the graph and the table, we conclude that

$$\lim_{x \to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3} \approx 0.294.$$

Example 2: Approximating the Value of a Limit

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x + 1, & x < 0, \\ -x^2 + 1, & x > 0. \end{cases}$$

Graphical Approximation. We graph $f(x)$ on an interval containing $x = 0$. The graph shows that from both the left and the right, the y-values approach 1. The open circle at $(0, 1)$ indicates that $f(0)$ is not defined.

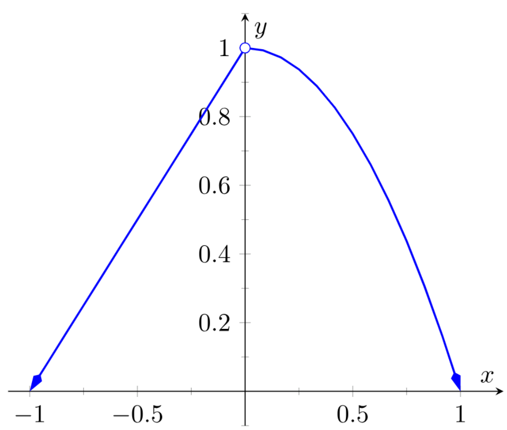


Figure 1.4: Graph of the piecewise function $f(x)$ near $x = 0$.

Numerical Approximation. We construct a table of values near $x = 0$.

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Since the values of $f(x)$ approach 1 from both sides, we conclude that

$$\lim_{x\to 0} f(x) = 1.$$

1.1.5 Definition of a Limit

Working definition (intuitive) We write

$$\lim_{x\to a} f(x) = L$$

if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a, using values on both sides of a, without requiring $x = a$.

A limit is about what the function is doing *around* $x = a$. It does not ask for $f(a)$, and $f(a)$ can be different from the limit.

Example (hole in the graph, limit still exists)

Estimate

$$\lim_{x\to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}.$$

You do *not* plug in $x = 2$ because the expression is undefined there. Instead, you look at values of x close to 2 from the left and right (table or graph). The values approach 4, so

$$\lim_{x\to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4.$$

Example (changing the function value does not change the limit)

Define

$$g(x) = \begin{cases} \dfrac{x^2 + 4x - 12}{x^2 - 2x}, & x \neq 2, \\ 6, & x = 2. \end{cases}$$

Even though $g(2) = 6$, the values of $g(x)$ near $x = 2$ still approach 4. So,

$$\lim_{x \to 2} g(x) = 4.$$

1.2 One-Sided Limits

Sometimes we are interested in what happens to a function as x approaches a value from only one side.

1.2.1 Right-Hand Limit:

We write

$$\lim_{x \to a^+} f(x) = L$$

to mean that as x approaches a from values greater than a (from the right), the values of $f(x)$ approach L.

1.2.2 Left-Hand Limit:

We write

$$\lim_{x \to a^-} f(x) = L$$

to mean that as x approaches a from values less than a (from the left), the values of $f(x)$ approach L.

Note: The limit

$$\lim_{x \to a} f(x)$$

exists if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

If these two one-sided limits are different, then the limit does not exist.

Example 1: A Jump Discontinuity

Let

$$f(x) = \begin{cases} 1, & x < 0, \\ 3, & x > 0. \end{cases}$$

Find the left-hand and right-hand limits as $x \to 0$.

Left-hand limit:

$$\lim_{x \to 0^-} f(x) = 1.$$

Right-hand limit:

$$\lim_{x \to 0^+} f(x) = 3.$$

Since

$$1 \neq 3,$$

the two one-sided limits are not equal.
Therefore,

$$\lim_{x \to 0} f(x)$$

does not exist.

Example 2: Piecewise Function with Equal One-Sided Limits

Let

$$g(x) = \begin{cases} x + 2, & x < 1, \\ 3x - 1, & x > 1. \end{cases}$$

Find $\lim_{x \to 1} g(x)$.

Left-hand limit:

$$\lim_{x \to 1^-} g(x) = 1 + 2 = 3.$$

Right-hand limit:

$$\lim_{x \to 1^+} g(x) = 3(1) - 1 = 2.$$

Since

$$3 \neq 2,$$

the limit does not exist.

Example 3: Equal One-Sided Limits

Let

$$h(x) = \begin{cases} x^2, & x < 2, \\ 4, & x = 2, \\ x^2, & x > 2. \end{cases}$$

Find $\lim_{x \to 2} h(x)$.

Left-hand limit:

$$\lim_{x \to 2^-} x^2 = 4.$$

Right-hand limit:

$$\lim_{x \to 2^+} x^2 = 4.$$

Since both one-sided limits are equal,

$$\lim_{x \to 2} h(x) = 4.$$

Notice that the limit exists even though the function is defined separately at $x = 2$.

1.3 Properties of Limits / Limit Laws

Limit laws help us evaluate limits algebraically without building tables or graphs, provided the individual limits exist.
Suppose

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M.$$

Then the following laws hold.

Limit Laws

Suppose

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M.$$

Then the following laws hold:

#	**Name**	**Law**
1	Constant Law	$\lim_{x \to a} c = c$
2	Identity Law	$\lim_{x \to a} x = a$
3	Sum Law	$\lim_{x \to a} [f(x) + g(x)] = L + M$
4	Difference Law	$\lim_{x \to a} [f(x) - g(x)] = L - M$
5	Constant Multiple Law	$\lim_{x \to a} [cf(x)] = cL$
6	Product Law	$\lim_{x \to a} [f(x)g(x)] = LM$
7	Quotient Law	$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
8	Power Law	$\lim_{x \to a} [f(x)]^n = L^n$
9	Root Law	$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

These laws justify direct substitution for polynomials and many other basic functions. Limit laws can only be used when the individual limits exist. In particular, the Quotient Law requires that

$$\lim_{x \to a} g(x) \neq 0.$$

Example 1: Using Limit Laws

Evaluate

$$\lim_{x\to 2}(3x^2 - 5x + 4).$$

Using the Sum and Constant Multiple Laws,

$$\lim_{x\to 2} 3x^2 = 3\lim_{x\to 2} x^2 = 3(2^2) = 12.$$

$$\lim_{x\to 2}(-5x) = -5\lim_{x\to 2} x = -5(2) = -10.$$

$$\lim_{x\to 2} 4 = 4.$$

Therefore,

$$\lim_{x\to 2}(3x^2 - 5x + 4) = 12 - 10 + 4 = 6.$$

Notice that this is simply direct substitution.

Example 2: A Rational Function

Evaluate

$$\lim_{x\to 3}\frac{x^2 - 1}{2x + 5}.$$

First compute numerator and denominator limits separately.

$$\lim_{x\to 3}(x^2 - 1) = 3^2 - 1 = 8.$$

$$\lim_{x\to 3}(2x + 5) = 2(3) + 5 = 11.$$

Since the denominator limit is not zero, we apply the Quotient Law.

$$\lim_{x\to 3}\frac{x^2 - 1}{2x + 5} = \frac{8}{11}.$$

Example 3: When the Quotient Law Cannot Be Used

Evaluate

$$\lim_{x\to 2}\frac{x^2 - 4}{x - 2}.$$

Direct substitution gives

$$\frac{2^2 - 4}{2 - 2} = \frac{0}{0}.$$

Since the denominator limit is zero, the Quotient Law does not apply.
The expression $\frac{0}{0}$ is called an indeterminate form. This type of limit requires algebraic manipulation (factoring) before applying limit laws.

1.4 Computing Limits

In many limits, direct substitution works immediately. However, sometimes substitution gives an expression such as

$$\frac{0}{0}$$

This is called an **indeterminate form**. It does not mean the limit is zero. It means we must simplify first.

1.4.1 Algebraic Simplification (Factoring and Canceling)

When substitution gives $\frac{0}{0}$, we often factor and cancel.

Example 1: Factoring

Evaluate

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

Direct substitution gives

$$\frac{2^2 - 4}{2 - 2} = \frac{0}{0}.$$

We first factor the numerator:

$$x^2 - 4 = (x - 2)(x + 2).$$

Then

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}.$$

For $x \neq 2$, we cancel:

$$= x + 2.$$

Now compute the limit:

$$\lim_{x \to 2} (x + 2) = 4.$$

$$\boxed{\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4}$$

1.4.2 Rationalizing

Sometimes radicals cause indeterminate forms.

Example 2: Rationalizing

Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}.$$

Substitution gives $\frac{0}{0}$. So we need to multiply it by the conjugate:

$$\frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}.$$

The numerator becomes

$$(x+1)-1 = x.$$

So we get

$$\frac{x}{x(\sqrt{x+1}+1)}.$$

Cancel x:

$$\frac{1}{\sqrt{x+1}+1}.$$

Now substitute:

$$\frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

$$\boxed{\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \frac{1}{2}}$$

1.4.3 The Squeeze Theorem

Sometimes a function is difficult to evaluate directly. Instead, we trap it between two easier functions. Suppose that for all x near c,

$$f(x) \leq h(x) \leq g(x),$$

and suppose

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L.$$

Then

$$\lim_{x \to c} h(x) = L.$$

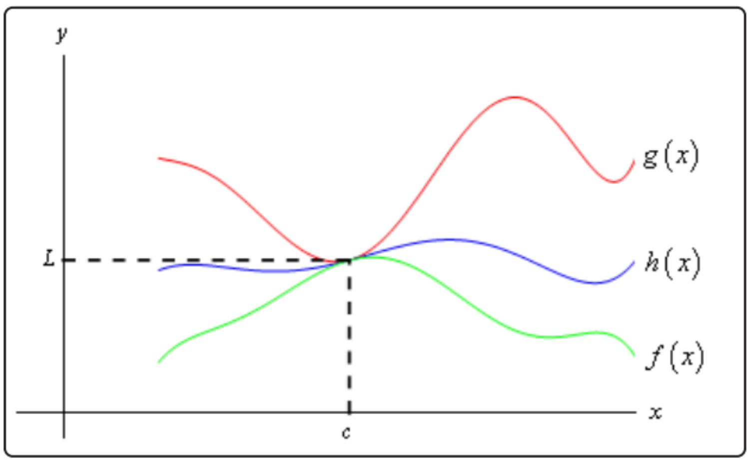


Figure 1.5: If $f(x) \leq h(x) \leq g(x)$ and both outer functions approach L, then $h(x)$ is squeezed to L.

Example 3: Proving $\lim_{x \to 0} \frac{\sin x}{x} = 1$

For x near 0, it can be shown that

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

We know

$$\lim_{x \to 0} \cos x = 1 \quad \text{and} \quad \lim_{x \to 0} 1 = 1.$$

Since both bounding functions approach 1, the Squeeze Theorem gives

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.}$$

This result is fundamental in calculus and will be used repeatedly.

1.5 Infinite Limits

So far, we have studied limits that approach a finite number. However, sometimes as x approaches a value, the function grows without bound. In such cases, we say the limit is infinite.

1.5.1 Definition

We write

$$\lim_{x \to a} f(x) = \infty$$

if the values of $f(x)$ increase without bound as x approaches a.
Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

if the values of $f(x)$ decrease without bound as x approaches a.

Important note !!

An infinite limit does not mean the limit exists as a real number. It means the function grows without bound.

1.5.2 One-Sided Infinite Limits

Often, infinite behavior happens from only one side.

$$\lim_{x \to a^-} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty.$$

We must always check left-hand and right-hand behavior separately.

Example 1: Basic Infinite Limit

Evaluate

$$\lim_{x \to 0} \frac{1}{x^2}.$$

As $x \to 0^-$, $x^2 \to 0^+$, so

$$\frac{1}{x^2} \to +\infty.$$

As $x \to 0^+$, $x^2 \to 0^+$, so again

$$\frac{1}{x^2} \to +\infty.$$

Therefore,

$$\boxed{\lim_{x \to 0} \frac{1}{x^2} = \infty.}$$

Example 2: Different One-Sided Behavior

Evaluate

$$\lim_{x\to 0} \frac{1}{x}.$$

As $x \to 0^+$,

$$\frac{1}{x} \to +\infty.$$

As $x \to 0^-$,

$$\frac{1}{x} \to -\infty.$$

Since the left-hand and right-hand limits are different,

$$\boxed{\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}}$$

1.5.3 Vertical Asymptotes

If

$$\lim_{x\to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \pm\infty,$$

then the line $x = a$ is called a **vertical asymptote**.

Example 3: Vertical Asymptote

Determine whether a vertical asymptote exists for

$$f(x) = \frac{1}{x-3}.$$

As $x \to 3^+$,

$$\frac{1}{x-3} \to +\infty.$$

As $x \to 3^-$,

$$\frac{1}{x-3} \to -\infty.$$

Therefore,

$$\boxed{x = 3 \text{ is a vertical asymptote.}}$$

1.6 Limits at Infinity

A limit at infinity describes the long-term behavior of a function. Instead of asking what happens as x approaches a specific number, we ask what happens as x becomes arbitrarily large in the positive or negative direction.

The notation

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x)$$

represents the value that $f(x)$ approaches as x increases without bound or decreases without bound.

If

$$\lim_{x\to\infty} f(x) = L,$$

then for sufficiently large values of x, the function values $f(x)$ can be made arbitrarily close to L.

Similarly, if

$$\lim_{x\to-\infty} f(x) = L,$$

then as x becomes very large in the negative direction, the function values approach L.

to describe the **end behavior** of a function.

1.6.1 Basic Idea

If

$$\lim_{x\to\infty} f(x) = L,$$

then as x becomes very large, the function values get closer and closer to L.

Horizontal Asymptote:

If

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L,$$

then the line

$$y = L$$

is called a **horizontal asymptote**.

Example 1: A Simple Rational Function

Evaluate

$$\lim_{x\to\infty} \frac{1}{x}.$$

As x becomes large,

$$\frac{1}{x} \to 0.$$

Therefore,

$$\boxed{\lim_{x\to\infty} \frac{1}{x} = 0}$$

So $y = 0$ is a horizontal asymptote.

1.6.2 Rational Functions and Leading Terms

For rational functions, limits at infinity depend on the degrees of numerator and denominator. Consider

$$f(x) = \frac{a_n x^n + \dots}{b_m x^m + \dots}.$$

The behavior depends on comparing n and m.

Case 1: Degree of Numerator < Degree of Denominator

$$\lim_{x \to \pm\infty} f(x) = 0.$$

Case 2: Degrees Equal

$$\lim_{x \to \pm\infty} f(x) = \frac{\text{leading coefficient of numerator}}{\text{leading coefficient of denominator}}.$$

Case 3: Degree of Numerator > Degree of Denominator

The limit is infinite (or does not exist as a finite number).

In summary, for

$$f(x) = \frac{a_n x^n + \dots}{b_m x^m + \dots},$$

- If $n < m$, limit is 0.
- If $n = m$, limit is $\frac{a_n}{b_m}$.
- If $n > m$, the function grows without bound.

Example 2: Equal Degrees

Evaluate

$$\lim_{x \to \infty} \frac{3x^2 + 5x - 1}{2x^2 - 7}.$$

To analyze the behavior as $x \to \infty$, divide every term in the numerator and denominator by x^2, the highest power of x in the denominator.

$$= \lim_{x \to \infty} \frac{\frac{3x^2}{x^2} + \frac{5x}{x^2} - \frac{1}{x^2}}{\frac{2x^2}{x^2} - \frac{7}{x^2}}.$$

Simplify each term:

$$= \lim_{x\to\infty} \frac{3 + \frac{5}{x} - \frac{1}{x^2}}{2 - \frac{7}{x^2}}.$$

Now evaluate the limit. As $x \to \infty$,

$$\frac{5}{x} \to 0, \quad \frac{1}{x^2} \to 0, \quad \frac{7}{x^2} \to 0.$$

Therefore,

$$= \frac{3 + 0 - 0}{2 - 0} = \frac{3}{2}.$$

$$\boxed{\lim_{x\to\infty} \frac{3x^2 + 5x - 1}{2x^2 - 7} = \frac{3}{2}}$$ Thus $y = \frac{3}{2}$ is a horizontal asymptote.

Example 3: Degree of Numerator Smaller

Evaluate

$$\lim_{x\to-\infty} \frac{5x + 1}{x^2 + 4}.$$

The numerator has degree 1. The denominator has degree 2.

Since numerator degree < denominator degree,

$$\boxed{\lim_{x\to-\infty} \frac{5x + 1}{x^2 + 4} = 0.}$$

Thus $y = 0$ is a horizontal asymptote.

1.7 Continuity

Continuity formalizes the idea that a graph has no breaks, jumps, or holes at a point. A function $f(x)$ is said to be **continuous at** $x = a$ if all three of the following conditions are satisfied:

1. $f(a)$ exists.
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$.

A function is continuous at $x = a$ if and only if

$$\lim_{x\to a} f(x) = f(a).$$

If any one of these three conditions fails, the function is not continuous at $x = a$.

Example 1: A Continuous Function

Determine whether

$$f(x) = x^2 - 3x + 1$$

is continuous at $x = 2$.

Polynomials are defined everywhere and are continuous for all real numbers.

We first compute:

$$f(2) = 2^2 - 3(2) + 1 = 4 - 6 + 1 = -1.$$

$$\lim_{x \to 2} (x^2 - 3x + 1) = -1.$$

Since

$$\lim_{x \to 2} f(x) = f(2),$$

the function is continuous at $x = 2$.

Example 2: A Removable Discontinuity

Determine whether

$$f(x) = \frac{x^2 - 4}{x - 2}$$

is continuous at $x = 2$.

First note that $f(2)$ is undefined. However, we previously showed that

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

Since $f(2)$ does not exist, the function is not continuous at $x = 2$. This is called a **removable discontinuity** (a hole in the graph).

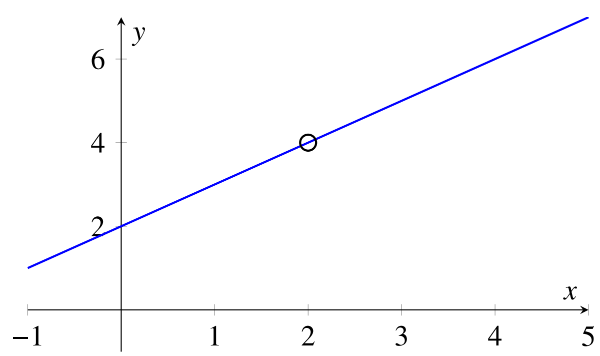


Figure 1.6: Graph of $f(x) = \frac{x^2 - 4}{x - 2}$. The graph is the line $y = x + 2$ with a hole at $x = 2$.

Example 3: A Jump Discontinuity

Let

$$g(x) = \begin{cases} 1, & x < 0, \\ 3, & x > 0. \end{cases}$$

We have

$$\lim_{x \to 0^-} g(x) = 1, \quad \lim_{x \to 0^+} g(x) = 3.$$

Since the one-sided limits are not equal, the limit at $x = 0$ does not exist.

Therefore, the function is not continuous at $x = 0$. This is called a **jump discontinuity**.

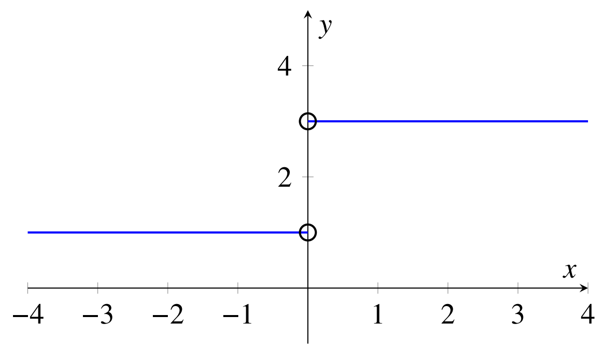


Figure 1.7: Graph of the piecewise function showing a jump at $x = 0$.

1.7.1 Continuity on an Interval

A function is continuous on an interval if it is continuous at every point in that interval. This means that for every number a in the interval,

$$\lim_{x \to a} f(x) = f(a).$$

For example, the function

$$f(x) = x^2$$

is continuous on $(-\infty, \infty)$ because it is continuous at every real number.

In contrast the funtion,

$$f(x) = \frac{1}{x}$$

is not continuous on $(-\infty, \infty)$, since it is undefined at $x = 0$. However, it is continuous on the intervals

$$(-\infty, 0) \quad \text{and} \quad (0, \infty).$$

1.7.2 Functions That Are Continuous Everywhere

Certain basic functions are continuous at every point in their domains.

Function Type	**Example**	**Where Continuous**
Polynomials	$f(x) = 3x^4 - 2x + 1$	All real numbers
Rational functions	$f(x) = \frac{x+1}{x-3}$	All real numbers except $x = 3$
Exponential functions	$f(x) = e^x$	All real numbers
Trigonometric functions	$f(x) = \sin x$	All real numbers
Root functions	$f(x) = \sqrt{x}$	$[0, \infty)$

Algebra of Continuous Functions

If f and g are continuous at $x = a$, then the following are also continuous at $x = a$:

$$f(x) + g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)} \quad (\text{if } g(a) \neq 0).$$

For Example:
Let

$$f(x) = x^2 \quad \text{and} \quad g(x) = \sin x.$$

Since both functions are continuous everywhere, the function

$$h(x) = x^2 \sin x$$

is also continuous everywhere.

1.8 Formal ε–δ Definition of a Limit

Definition 1: (Finite Limit at a Finite Point)

Let $f(x)$ be defined on an interval containing $x = a$, except possibly at $x = a$ itself.
We say that

$$\lim_{x\to a} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Understanding the Definition

- ε measures how close we want $f(x)$ to be to L.
- δ measures how close x must be to a.
- The condition $0 < |x - a| < \delta$ means we are close to a, but not equal to a.
- The inequality $|f(x) - L| < \varepsilon$ means the function values are within an ε-band around L.

Example 1

Prove $\lim_{x\to 0} x^2 = 0$

We must show: For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x^2 - 0| < \varepsilon \quad \text{whenever} \quad 0 < |x| < \delta.$$

Since

$$|x^2| = x^2,$$

we want

$$x^2 < \varepsilon.$$

This happens whenever

$$|x| < \sqrt{\varepsilon}.$$

So we choose

$$\delta = \sqrt{\varepsilon}.$$

Then if $0 < |x| < \delta$,

$$|x^2| < (\sqrt{\varepsilon})^2 = \varepsilon.$$

Therefore, $\lim\limits_{x\to 0} x^2 = 0$.

Example 2

Prove $\lim_{x\to 2}(5x-4)=6$

We want to prove using Definition 1 that

$$\lim_{x\to 2}(5x-4)=6.$$

We start with the expression

$$|f(x)-L|=|(5x-4)-6|.$$

Simplify:

$$|(5x-4)-6|=|5x-10|=5|x-2|.$$

Then we make this less than ε
We want

$$5|x-2|<\varepsilon.$$

Divide both sides by 5:

$$|x-2|<\frac{\varepsilon}{5}.$$

Choose δs

$$\delta=\frac{\varepsilon}{5}.$$

To verify
If $0<|x-2|<\delta$, then

$$|f(x)-6|=5|x-2|<5\left(\frac{\varepsilon}{5}\right)=\varepsilon.$$

Therefore, by Definition 1, $\lim_{x\to 2}(5x-4)=6$.

Definition 2: (Right-Hand Limit)

We say

$$\lim_{x\to a^+} f(x)=L$$

if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|f(x)-L|<\varepsilon \quad \text{whenever} \quad 0<x-a<\delta.$$

Definition 3: (Left-Hand Limit)

We say

$$\lim_{x\to a^-} f(x)=L$$

if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|f(x)-L|<\varepsilon \quad \text{whenever} \quad -\delta<x-a<0.$$

Definition 4: (Limit Equals ∞)

We say

$$\lim_{x \to a} f(x) = \infty$$

if for every number $M > 0$, there exists $\delta > 0$ such that

$$f(x) > M \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Example 5:

Prove $\lim_{x\to 0} \frac{1}{x^2} = \infty$

We use the definition of an infinite limit.We must show that for every $M > 0$, there exists $\delta > 0$ such that

$$\frac{1}{x^2} > M \quad \text{whenever} \quad 0 < |x| < \delta.$$

First we solve the inequality. We want

$$\frac{1}{x^2} > M.$$

Multiply both sides by x^2 (which is positive): $1 > Mx^2 \implies \frac{1}{M} > x^2 \implies |x| < \frac{1}{\sqrt{M}}$

Then choose δ

$$\delta = \frac{1}{\sqrt{M}}.$$

To verify, if $0 < |x| < \delta$, then

$$|x| < \frac{1}{\sqrt{M}}.$$

Squaring both sides:

$$x^2 < \frac{1}{M}.$$

Taking reciprocals (since both sides are positive):

$$\frac{1}{x^2} > M.$$

Therefore, by the definition of an infinite limit, $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$.

Definition 5: (Limit Equals $-\infty$)

We say
$$\lim_{x\to a} f(x) = -\infty$$
if for every number $N < 0$, there exists $\delta > 0$ such that
$$f(x) < N \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Definition 6: (Limit at $+\infty$)

We say
$$\lim_{x\to \infty} f(x) = L$$
if for every $\varepsilon > 0$ there exists $M > 0$ such that
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

Definition 7: (Limit as $x \to -\infty$)

We say
$$\lim_{x\to -\infty} f(x) = L$$
if for every $\varepsilon > 0$, there exists $N < 0$ such that
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

Definition 8: (Limit at Infinity Equals ∞)

We say
$$\lim_{x\to \infty} f(x) = \infty$$
if for every $N > 0$ there exists $M > 0$ such that
$$f(x) > N \quad \text{whenever} \quad x > M.$$

Definition 9: (Continuity)

A function $f(x)$ is continuous at $x = a$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that
$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Chapter 2: Derivatives

2.1 Definition of the Derivative

The derivative of $f(x)$ with respect to x is the function $f'(x)$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We read $f'(x)$ as "f prime of x."

The derivative represents the instantaneous rate of change of the function and the slope of the tangent line at x.

Example 1

$$f(x) = 2x^2 - 16x + 35$$

Plug into the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

First compute $f(x+h)$:

$$f(x+h) = 2(x+h)^2 - 16(x+h) + 35.$$

Expand:

$$= 2(x^2 + 2xh + h^2) - 16x - 16h + 35.$$

$$= 2x^2 + 4xh + 2h^2 - 16x - 16h + 35.$$

Now subtract $f(x)$:

$$f(x+h) - f(x) = (2x^2 + 4xh + 2h^2 - 16x - 16h + 35) - (2x^2 - 16x + 35).$$

Distribute the negative:

$$= 2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35.$$

$$= 4xh + 2h^2 - 16h.$$

Factor h:

$$= h(4x + 2h - 16).$$

Now divide by h:

$$\frac{h(4x+2h-16)}{h} = 4x+2h-16.$$

Take the limit:

$$f'(x) = \lim_{h\to 0}(4x+2h-16) = 4x-16.$$

2.1.1 Differentiablity

A function $f(x)$ is called **differentiable at** $x = a$ if $f'(a)$ exists. It is called **differentiable on an interval** if the derivative exists at every point in the interval.

Theorem:
If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$. The converse is not true. A function may be continuous at a point but not differentiable there.

Example

$$f(x) = |x| \text{ at } x = 0$$

Consider the function

$$f(x) = |x|.$$

First observe that

$$\lim_{x\to 0}|x| = 0 = f(0),$$

so $f(x)$ is continuous at $x = 0$.
Now compute the derivative using the definition.
For $x > 0$,

$$|x| = x, \quad \text{so} \quad f'(x) = 1.$$

For $x < 0$,

$$|x| = -x, \quad \text{so} \quad f'(x) = -1.$$

Thus,

$$\lim_{x\to 0^-}\frac{|x|-0}{x} = -1, \quad \lim_{x\to 0^+}\frac{|x|-0}{x} = 1.$$

Since the one-sided limits are not equal, the derivative at $x = 0$ does not exist. Therefore, $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

2.1.2 Alternate Notation

Given $y = f(x)$, the following notations are equivalent and represent the derivative of $f(x)$ with respect to x:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \frac{d}{dx}(y).$$

To evaluate the derivative at $x = a$, the following notations are equivalent:

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}.$$

In some contexts, the variable may be omitted for simplicity:

$$f'(x) = f'.$$

2.2 Basic Derivative Rules

The definition of the derivative is fundamental, but using it every time can require a lot of algebra. For that reason, we develop derivative rules that allow us to compute derivatives more quickly.
We begin with one of the most important rules in differential calculus: the **power rule**.
To motivate the rule, we first compute the derivatives of x^2 and x^3 directly from the definition.

Example 1

Determine the derivative of $f(x) = x^2$
Using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Since $f(x) = x^2$,

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

Expand $(x+h)^2$:

$$(x+h)^2 = x^2 + 2xh + h^2.$$

Substitute this back into the limit:

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}.$$

Simplify the numerator:

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h}.$$

Factor out h:

$$f'(x) = \lim_{h \to 0} \frac{h(2x+h)}{h}.$$

Cancel h:

$$f'(x) = \lim_{h \to 0} (2x + h).$$

Now let $h \to 0$:

$$f'(x) = 2x.$$

Therefore,

$$\boxed{\frac{d}{dx}(x^2) = 2x}.$$

Example 2

Determine the derivative of $f(x) = x^3$

Again, use the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Since $f(x) = x^3$,

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}.$$

Expand $(x+h)^3$:

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3.$$

Substitute:

$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}.$$

Simplify the numerator:

$$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}.$$

Factor out h:

$$f'(x) = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}.$$

Cancel h:

$$f'(x) = \lim_{h \to 0} (3x^2 + 3xh + h^2).$$

Now let $h \to 0$:

$$f'(x) = 3x^2.$$

Therefore,

$$\boxed{\frac{d}{dx}(x^3) = 3x^2}.$$

2.2.1 The Power Rule

The previous two examples show a general pattern. The derivative of a power of x is obtained by bringing the exponent down in front and then subtracting 1 from the exponent.

Power Rule:
Let $n \in \mathbb{Z}$. Then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

This rule works for positive integers, negative integers, and zero, as long as the expression is defined.

Example 3: Use the Power Rule

Determine the derivative of each function.

(a) $f(x) = x^5$

Using the power rule,

$$f'(x) = 5x^4.$$

(b) $g(x) = x^9$

$$g'(x) = 9x^8.$$

(c) $p(x) = \frac{1}{x}$

First rewrite the function using exponents:

$$p(x) = x^{-1}.$$

Now apply the power rule:

$$p'(x) = -1x^{-2} = -\frac{1}{x^2}.$$

(d) $g(x) = \frac{1}{x^3}$

Rewrite:

$$g(x) = x^{-3}.$$

Then

$$g'(x) = -3x^{-4} = -\frac{3}{x^4}.$$

2.2.2 The Constant rule

The power rule also helps explain the derivative of a constant. If c is a constant, then its graph is a horizontal line, and the slope of a horizontal line is 0. This leads to the constant rule:

Constant Rule:

If c is a constant, then

$$\frac{d}{dx}(c) = 0.$$

Example 1

Determine the derivative of $f(x) = 4$

Since 4 is a constant,

$$\frac{d}{dx}(4) = 0.$$

Therefore,

$$\boxed{\frac{d}{dx}(4) = 0}.$$

2.2.3 Sum and Difference Rules

The power rule and constant rule let us differentiate single terms such as x^5 or 4. However, many functions are made up of several terms added or subtracted together. The sum and difference rules allow us to differentiate such functions term by term.

Sum/Difference Rule:

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

and

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)].$$

Proof of the Sum Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then, by the definition of the derivative,

$$\frac{d}{dx}[f(x)] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and

$$\frac{d}{dx}[g(x)] = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

Now consider the derivative of the sum:

$$\frac{d}{dx}[f(x) + g(x)] = \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}.$$

Remove the brackets in the numerator:

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}.$$

Group the f-terms and g-terms together:

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}.$$

Split the fraction:

$$= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

Apply the limit law for sums:

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}.$$

Recognize each limit as a derivative:

$$= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)].$$

Therefore,

$$\boxed{\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]}$$

The proof of the difference rule is similar.

Examples

Determine the derivative of each function.

(a) $f(x) = x^3 - x^2 + x + 7$

Apply the sum and difference rules term by term:

$$f'(x) = \frac{d}{dx}[x^3 - x^2 + x + 7]$$

$$= \frac{d}{dx}[x^3] - \frac{d}{dx}[x^2] + \frac{d}{dx}[x] + \frac{d}{dx}[7].$$

Now use the power rule and constant rule:

$$= 3x^2 - 2x + 1 + 0.$$

Therefore,

$$\boxed{f'(x) = 3x^2 - 2x + 1.}$$

(b) $g(x) = x^4 - \dfrac{1}{x^3} - 6$
First rewrite the function using exponents:

$$g(x) = x^4 - x^{-3} - 6.$$

Now differentiate term by term:

$$g'(x) = \frac{d}{dx}[x^4 - x^{-3} - 6]$$
$$= \frac{d}{dx}[x^4] - \frac{d}{dx}[x^{-3}] - \frac{d}{dx}[6].$$

Apply the power rule and constant rule:

$$= 4x^3 - \left(-3x^{-4}\right) - 0.$$

Simplify:

$$= 4x^3 + 3x^{-4}.$$

Rewrite with positive exponents if you want:

$$\boxed{g'(x) = 4x^3 + \frac{3}{x^4}}$$

2.2.4 Constant Multiple Rule

Often a function is multiplied by a constant. The constant multiple rule tells us that the constant can be factored out when taking the derivative.

Constant Multiple Rule:
Let $f(x)$ be a differentiable function and let c be a constant or any real number. Then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)].$$

Proof of the Constant Multiple Rule

Let $f(x)$ be differentiable. Using the definition of the derivative,

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h}.$$

Factor out the constant c:

$$= \lim_{h\to 0} \frac{c[f(x+h) - f(x)]}{h}.$$

Move the constant outside the limit:

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Recognize the definition of the derivative:

$$= c \frac{d}{dx}[f(x)].$$

Therefore,

$$\boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)].}$$

Example 1

Determine the derivative of each function

(a) $f(x) = 5x^3$

$$f'(x) = \frac{d}{dx}[5x^3]$$

Apply the constant multiple rule:

$$= 5\frac{d}{dx}[x^3]$$

Apply the power rule:

$$= 5(3x^2)$$

$$\boxed{f'(x) = 15x^2}$$

(b) $g(x) = -3x^4 - \dfrac{5}{x^2}$

First rewrite using exponents:

$$g(x) = -3x^4 - 5x^{-2}$$

Now differentiate term by term:

$$g'(x) = \frac{d}{dx}[-3x^4] - \frac{d}{dx}[5x^{-2}]$$

Apply the constant multiple rule and power rule:

$$= -3(4x^3) - 5(-2x^{-3})$$

$$= -12x^3 + 10x^{-3}$$

Rewrite with positive exponents:

$$\boxed{g'(x) = -12x^3 + \frac{10}{x^3}}$$

Example 2

Compute the derivative of each function

(a) $f(x) = 3x^2 - 6x + 9$

$$f'(x) = 3\frac{d}{dx}[x^2] - 6\frac{d}{dx}[x] + \frac{d}{dx}[9]$$
$$= 3(2x) - 6(1) + 0$$
$$\boxed{f'(x) = 6x - 6}$$

(b) $g(x) = 8 - 16x^3$

$$g'(x) = \frac{d}{dx}[8] - 16\frac{d}{dx}[x^3]$$
$$= 0 - 16(3x^2)$$
$$\boxed{g'(x) = -48x^2}$$

(c) $p(x) = \frac{x^2}{6} - 7x + 1$

$$p'(x) = \frac{1}{6}\frac{d}{dx}[x^2] - 7\frac{d}{dx}[x] + \frac{d}{dx}[1]$$
$$= \frac{1}{6}(2x) - 7 + 0$$
$$\boxed{p'(x) = \frac{x}{3} - 7}$$

(d) $q(x) = \frac{4}{x^3} - 6$

$$q(x) = 4x^{-3} - 6$$
$$q'(x) = 4\frac{d}{dx}[x^{-3}] - \frac{d}{dx}[6]$$
$$= 4(-3x^{-4}) - 0$$
$$= -12x^{-4}$$
$$\boxed{q'(x) = -\frac{12}{x^4}}$$

(e) $r(x) = 7x - \frac{2}{x}$

$$r(x) = 7x - 2x^{-1} \implies r'(x) = 7\frac{d}{dx}[x] - 2\frac{d}{dx}[x^{-1}] = \boxed{r'(x) = 7 + 2x^{-2} = 7 + \frac{2}{x^2}}$$

Example 3

After a football is punted, its height h (in meters) above the ground after t seconds can be modeled by

$$h(t) = -4.9t^2 + 21t + 0.45.$$

1. When does the football reach its maximum height?

2. What is the football's maximum height?

Solution

The maximum height occurs when the slope of the tangent line is zero.
First, we compute the derivative of the height function.

$$h(t) = -4.9t^2 + 21t + 0.45$$

$$h'(t) = -9.8t + 21$$

The, we set the derivative equal to zero.

$$-9.8t + 21 = 0$$

Solve for t:

$$21 = 9.8t$$
$$t = \frac{21}{9.8}$$
$$t \approx 2.14$$

So, the football reaches its maximum height at approximately

$$\boxed{t \approx 2.14 \text{ seconds}}$$

Now we need to substitute $t = 2.14$ into the height function.

$$h(2.14) = -4.9(2.14)^2 + 21(2.14) + 0.45$$

$$\approx -4.9(4.58) + 44.94 + 0.45$$

$$\approx -22.44 + 44.94 + 0.45$$

$$\approx 22.95$$

Therefore, the maximum height is approximately

$$\boxed{22.95 \text{ meters}}$$

2.2.5 Differentiability and Continuity

A natural question is how differentiability (having a derivative) is related to continuity.

Theorem 2.19
If a function $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is continuous at $x = a$.

Recall the definitions involved in this theorem.
A function $f(x)$ is **differentiable at** $x = a$ if the following limit exists:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Equivalently,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.
A function $f(x)$ is **continuous at** $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Proof:

Assume that $f(x)$ is differentiable at $x = a$. Then the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.
We can rewrite $f(x) - f(a)$ as

$$f(x) - f(a) = (x - a)\left(\frac{f(x) - f(a)}{x - a}\right).$$

Now take the limit as $x \to a$:

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} (x - a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Since

$$\lim_{x \to a} (x - a) = 0$$

and the derivative limit exists and is finite, the product becomes

$$0 \cdot f'(a) = 0.$$

Thus,

$$\lim_{x \to a} [f(x) - f(a)] = 0.$$

This implies

$$\lim_{x \to a} f(x) - f(a) = 0.$$

Therefore,

$$\lim_{x \to a} f(x) = f(a).$$

So $f(x)$ is continuous at $x = a$.

Differentiability implies continuity.

2.3 Derivative of Exponential and Logarithmic Functions

Exponential and logarithmic functions have their own derivative rules. These are very important because they appear often in growth models, decay models, and rate-of-change problems.
A special exponential function is e^x.

Derivative of e^x

$$\frac{d}{dx}[e^x] = e^x$$

In general, for an exponential function with base a, we use the following rule.

Theorem 2.21: Derivative of Exponential Functions
Let $a \in \mathbb{R}$, with $a > 0$ and $a \neq 1$. Then

$$\frac{d}{dx}[a^x] = (\ln a)a^x.$$

Example 2.22: Calculate the derivative of each function

(a) $f(x) = 3 \cdot 2^x - 7e^x$
Differentiate term by term:

$$f'(x) = 3\frac{d}{dx}[2^x] - 7\frac{d}{dx}[e^x]$$

Apply the exponential derivative rules:

$$= 3[(\ln 2)2^x] - 7e^x \quad \Longrightarrow \quad \boxed{f'(x) = (3\ln 2)2^x - 7e^x}$$

(b) $g(x) = 4 \cdot 3^x + 8 \cdot 5^x$
Differentiate term by term:

$$g'(x) = 4\frac{d}{dx}[3^x] + 8\frac{d}{dx}[5^x]$$

Apply the exponential derivative rule:

$$= 4(\ln 3)3^x + 8(\ln 5)5^x \quad \Longrightarrow \quad \boxed{g'(x) = (4\ln 3)3^x + (8\ln 5)5^x}$$

Example 2.23

Food bank usage in Britain has grown dramatically over the past decade. The number of users, in thousands, of the largest food bank is estimated to be

$$N(t) = 1.3(2.25)^t,$$

where t is the number of years since 2006.

1. The number of food bank users in 2008.
2. The rate of change in food bank users in 2008.

Solution:
Since t is the number of years since 2006, the year 2008 corresponds to

$$t = 2.$$

$$N(2) = 1.3(2.25)^2 \quad = 1.3(5.0625) \quad \Longrightarrow \quad \boxed{6.58125}$$

Because $N(t)$ is measured in thousands, this means $\boxed{\text{6.58125 thousand users}}$ which is approximately $\boxed{\text{6581 users}}$.

– > Rate of change in food bank users in 2008
First differentiate:

$$N'(t) = 1.3\frac{d}{dt}\left[(2.25)^t\right]$$
$$= 1.3(\ln 2.25)(2.25)^t$$

Now we evaluate at $t = 2$:

$$N'(2) = 1.3(\ln 2.25)(2.25)^2 \quad \Longrightarrow \quad \boxed{\approx 5.3369}$$

Because the output is in thousands of users per year, this means $\boxed{\text{5.3369344 thousand users per year}}$ which is approximately $\boxed{\text{5337 users per year}}$.

Let us now look at another exponential example involving a comparison of rates.

Example 2.24

The number of people infected with the flu in a certain city is given by

$$f(t) = 0.5e^t + t^2$$

(in hundreds), with t being the time (in days) since the flu season began. How many times higher is the rate of spread of the flu on day 6 than on day 2?

Solution:

The rate of spread is the derivative, so first compute $f'(t)$:

$$f'(t) = 0.5e^t + 2t$$

Now evaluate at $t = 6$ and $t = 2$:

$$f'(6) = 0.5e^6 + 2(6)$$

$$f'(2) = 0.5e^2 + 2(2)$$

The question asks how many times higher the rate is on day 6 than on day 2, so we compute the ratio:

$$\frac{f'(6)}{f'(2)} = \frac{0.5e^6 + 12}{0.5e^2 + 4} \implies \frac{213.7}{7.69} \approx \boxed{27.8}$$

So, the rate of spread on day 6 is about $\boxed{\text{28 times higher}}$ than the rate on day 2.

Now we turn to logarithmic functions. The natural logarithm function is the inverse of e^x. Its derivative is

Derivative of $\ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

More generally, if $f(x) = \log_b x$, then

Derivative of $\log_b x$

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$$

Example 2.26: Calculate the derivatives of the following functions

(a) $f(x) = 5\ln x$

$$f'(x) = 5\frac{d}{dx}[\ln x]$$

$$= 5\left(\frac{1}{x}\right) \implies \boxed{f'(x) = \frac{5}{x}}$$

(b) $g(x) = 7\log_3 x$

$$g'(x) = 7\frac{d}{dx}[\log_3 x]$$

$$= 7\left(\frac{1}{x\ln 3}\right) \implies \boxed{g'(x) = \frac{7}{x\ln 3}}$$

(c) $q(x) = \log_5 x - \log_2 x$

Differentiate term by term:

$$q'(x) = \frac{1}{x\ln 5} - \frac{1}{x\ln 2} \implies \boxed{q'(x) = \frac{1}{x\ln 5} - \frac{1}{x\ln 2}}$$

(d) $r(x) = \log_7 x + \ln x$

$$r'(x) = \frac{1}{x\ln 7} + \frac{1}{x} \implies \boxed{r'(x) = \frac{1}{x\ln 7} + \frac{1}{x}}$$

(e) $h(x) = \log_2 x^3$

Treat x^3 as the exponent on the logarithm expression:

$$h'(x) = 3 \cdot \frac{1}{x\ln 2} \implies \boxed{h'(x) = \frac{3}{x\ln 2}}$$

(f) $p(x) = \ln\sqrt{x}$

Rewrite first:

$$p(x) = \ln(x^{1/2})$$

Differentiate:

$$p'(x) = \frac{1}{2}\left(\frac{1}{x}\right) \implies \boxed{p'(x) = \frac{1}{2x}}$$

Example 2.27: Find the equation of the tangent line

Find the equation of the tangent line to the graph of

$$f(x) = \ln x$$

at $x = 4$.

Solution: First find the point on the graph.

$$y = f(4) = \ln 4$$

So the point is

$$(4, \ln 4).$$

Now find the slope.

$$f'(x) = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

Evaluate at $x = 4$:

$$f'(4) = \frac{1}{4}$$

So the slope is

$$m = \frac{1}{4}.$$

We use the slope–intercept form $y = mx + b$.

Substitute the point $(4, \ln 4)$:

$$\ln 4 = \frac{1}{4}(4) + b$$

$$\ln 4 = 1 + b$$

$$b = \ln 4 - 1$$

Therefore the equation of the tangent line is

$$\boxed{y = \frac{1}{4}x + \ln 4 - 1}$$

Example 2.28

A Cessna plane takes off from an airport at sea level and its altitude (in feet) at time t (in minutes) is given by

$$h(t) = 2000 \ln t.$$

Find the rate of the initial climb 3 minutes after take-off.

Solution: The rate of climb is the derivative.

$$h'(t) = 2000\frac{d}{dt}[\ln t] \implies 2000\left(\frac{1}{t}\right) \implies \boxed{\frac{2000}{t}}$$

Now evaluate at $t = 3$:

$$h'(3) = \frac{2000}{3}$$

$$\approx 666.67 \implies \boxed{666.67 \text{ ft/min}}$$

2.4 Product and Quotient Rules

Sometimes a function is the product of two functions. In this case we cannot simply differentiate each term separately. Instead we use the **product rule**.

Theorem 2.29: Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)].$$

Proof:
Start with the definition of the derivative.

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Add and subtract $f(x+h)g(x)$ inside the numerator.

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

Group the terms.

$$= \lim_{h\to 0}\left[\frac{f(x+h)(g(x+h) - g(x))}{h} + \frac{g(x)(f(x+h) - f(x))}{h}\right]$$

Separate the limits.

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h}$$

Take the limits.

$$= f(x)g'(x) + f'(x)g(x) \qquad \implies \qquad \boxed{\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)}$$

Example 2.30: Calculate the derivative of each function

(a) $f(x) = e^x x^3$

Apply the product rule.

$$f'(x) = \frac{d}{dx}[e^x] \cdot x^3 + e^x \cdot \frac{d}{dx}[x^3]$$
$$= e^x x^3 + e^x(3x^2)$$

Factor $e^x x^2$.

$$f'(x) = e^x x^2(x+3) \implies \boxed{f'(x) = e^x x^2(x+3)}$$

(b) $g(x) = (4-2x)(3x+1)$

Apply the product rule.

$$g'(x) = \frac{d}{dx}[4-2x](3x+1) + (4-2x)\frac{d}{dx}[3x+1]$$

$$(-2)(3x+1) + (4-2x)(3) = -6x - 2 + 12 - 6x = -12x + 10 \implies \boxed{g'(x) = -12x + 10}$$

—

Now consider the derivative of a quotient of two functions. **Quotient Rule**

Theorem 2.31: Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{(g(x))^2}$$

(*Suggested by Dr. Bouchat*) A way to remember this rule is

$$\frac{d}{dx}\left[\frac{\text{Hi}}{\text{Ho}}\right] = \frac{\text{Ho} \cdot d(\text{Hi}) - \text{Hi} \cdot d(\text{Ho})}{\text{Ho}^2}$$

—

Example 2.32

Calculate the derivative of f(x) in two ways: (1) by rewriting the expression and using the previous rules and (2) by using the quotient rule.

$$f(x) = \frac{3x^2 - 4x + 1}{x}$$

1. Rewrite the expression

$$f(x) = 3x - 4 + x^{-1}$$

Differentiate.

$$f'(x) = 3 - x^{-2} \implies 3 - \frac{1}{x^2} \implies \boxed{\frac{3x^2 - 1}{x^2}}$$

2. Use the quotient rule

$$f'(x) = \frac{(6x - 4)x - (3x^2 - 4x + 1)(1)}{x^2}$$

Simplify.

$$\frac{6x^2 - 4x - 3x^2 + 4x - 1}{x^2} \implies \boxed{\frac{3x^2 - 1}{x^2}}$$

Both methods give the same result.

$$\boxed{f'(x) = \frac{3x^2 - 1}{x^2}}$$

—

Example 2.33: Tangent Line

Find the equation of the tangent line to the graph of $y = f(x)$ at the indicated value. **1.** $f(x) = \frac{e^x}{x+1}$ at $x = 0$ First find the point.

$$f(0) = \frac{e^0}{0 + 1} = 1$$

Point:

$$(0, 1)$$

Now find the derivative using the quotient rule.

$$f'(x) = \frac{e^x(x + 1) - e^x(1)}{(x + 1)^2}$$

Factor e^x.

$$f'(x) = \frac{e^x[(x + 1) - 1]}{(x + 1)^2} \implies \boxed{f'(x) = \frac{xe^x}{(x + 1)^2}}$$

Now evaluate at $x = 0$.

$$f'(0) = \frac{e^0 \cdot 0}{1^2} = 0$$

Slope:

$$m = 0$$

The tangent line is horizontal.

$$\boxed{y = 1}$$

2. $f(x) = (4x - 1)(3x + 4)$ at $x = -3$
We first compute the derivative.

$$f'(x) = \frac{d}{dx}[4x - 1](3x + 4) + (4x - 1)\frac{d}{dx}[3x + 4]$$
$$= 4(3x + 4) + (4x - 1)(3)$$
$$= 12x + 16 + 12x - 3 \implies 24x + 13$$

Now evaluate the slope.

$$f'(-3) = 24(-3) + 13 \implies -72 + 13 = -59$$

Now find the point.

$$f(-3) = (4(-3) - 1)(3(-3) + 4)$$
$$= (-13)(-5) = 65$$

Point:

$$(-3, 65)$$

Use point slope form.

$$y - 65 = -59(x + 3) \implies \boxed{y = -59x - 112}$$

2.5 Chain rule

Suppose we want to take the derivative of the function

$$p(x) = (3x^2 - 5x + 2)^8$$

This function is a **composition of two functions**. That means a function is applied inside another function.

$$p(x) = f(g(x)) \quad \text{where} \quad f(x) = x^8 \quad \text{and} \quad g(x) = 3x^2 - 5x + 2$$

The chain rule says:

Rate of change of composite function = Rate of change of outside function×Rate of change of inside function

—

Theorem 2.35 (Chain Rule)

Suppose $y = f(g(x))$ is a composition of the differentiable functions $f(x)$ and $g(x)$. Let

$$z = g(x)$$

Then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

For example

Let

$$p(x) = (3x^2 - 5x + 2)^8$$

Let

$$z = 3x^2 - 5x + 2$$

so

$$p(x) = z^8$$

Now apply the chain rule.

$$\begin{aligned} p'(x) &= \frac{d}{dz}[z^8] \cdot \frac{d}{dx}[3x^2 - 5x + 2] \\ &= 8z^7(6x - 5) \end{aligned}$$

Substitute back z.

$$\boxed{p'(x) = 8(3x^2 - 5x + 2)^7(6x - 5)}$$

For a function $y = f(x)$, we can write

$$f'(x) = y' = \frac{d}{dx}[f(x)] = \frac{d}{dx}[y] = \frac{dy}{dx}$$

If the variable is z, we write

$$f'(z) = y' = \frac{d}{dz}[f(z)] = \frac{d}{dz}[y] = \frac{dy}{dz}$$

—

Example 2.36

Calculate the derivative of each function.

1. $f(x) = (4x-6)^7$

$$\text{Let} \qquad z = 4x-6 \qquad \text{and} \qquad f(x) = z^7$$

$$f'(x) = \frac{d}{dz}[z^7] \cdot \frac{d}{dx}[4x-6]$$

$$= 7z^6(4) \implies \boxed{28z^6}$$

Substitute back.

$$\boxed{f'(x) = 28(4x-6)^6}$$

2. $f(x) = \sqrt{3x-1}$

Rewrite:

$$f(x) = (3x-1)^{1/2}$$

$$\text{Let} \qquad z = 3x-1 \qquad \text{and} \qquad f(x) = z^{1/2}$$

$$f'(x) = \frac{d}{dz}[z^{1/2}] \cdot \frac{d}{dx}[3x-1]$$

$$= \frac{1}{2}z^{-1/2}(3) \qquad \implies \qquad \boxed{= \frac{3}{2}z^{-1/2}}$$

Substitute back.

$$\boxed{f'(x) = \frac{3}{2}(3x-1)^{-1/2}}$$

—

3. $f(x) = e^{3x-1}$

$$\text{Let} \qquad z = 3x-1 \qquad \text{and} \qquad f(x) = e^z$$

$$f'(x) = \frac{d}{dz}[e^z] \cdot \frac{d}{dx}[3x-1]$$

$$= e^z(3)$$

$$\boxed{f'(x) = 3e^{3x-1}}$$

—

Example 2.37

Calculate the derivative of each function.

1. $f(x) = (3x^2 - 9x + 10)^{20}$

$$\text{Let} \qquad z = 3x^2 - 9x + 10 \qquad \text{and} \qquad f(x) = z^{20}$$

$$f'(x) = \frac{d}{dz}[z^{20}] \cdot \frac{d}{dx}[3x^2 - 9x + 10]$$

$$= 20z^{19}(6x - 9) \quad \Longrightarrow \quad \boxed{f'(x) = 20(3x^2 - 9x + 10)^{19}(6x - 9)}$$

—

2. $f(x) = \sqrt[3]{5 - 4x}$

Rewrite

$$f(x) = (5 - 4x)^{1/3}$$

$$\text{Let} \qquad z = 5 - 4x \qquad \text{and} \qquad f(x) = z^{1/3}$$

$$f'(x) = \frac{d}{dz}[z^{1/3}] \cdot \frac{d}{dx}[5 - 4x]$$

$$= \frac{1}{3}z^{-2/3}(-4) \quad \Longrightarrow \quad \boxed{f'(x) = -\frac{4}{3}(5 - 4x)^{-2/3}}$$

—

3. $f(x) = e^{x^2 - x - 1}$

Let

$$z = x^2 - x - 1$$

$$f'(x) = \frac{d}{dz}[e^z] \cdot \frac{d}{dx}[x^2 - x - 1]$$

$$= e^z(2x - 1) \quad \Longrightarrow \quad \boxed{f'(x) = e^{x^2 - x - 1}(2x - 1)}$$

—

Example 2.38

The population of a city (in thousands) is

$$P(t) = 65e^{0.021t}$$

Find the rate of change in the year 2010. Since t is years after 2000,

$$t = 10$$

First we need to find the derivative.

$$P'(t) = 65e^{0.021t}\frac{d}{dt}[0.021t]$$

$$= 65e^{0.021t}(0.021) \implies 1.365e^{0.021t}$$

Evaluate at $t = 10$.

$$P'(10) = 1.365e^{0.21} \implies \boxed{P'(10) \approx 1.683}$$

So the population is increasing at approximately

$$\boxed{\text{1684 people per year}}$$

—

Example 2.40

The position function of a freight train is given by

$$s(t) = 100(t+1)^{-2}$$

with s in meters and t in seconds. Determine the train's velocity at 6 seconds.
Velocity is the derivative.

$$s'(t) = 100(-2)(t+1)^{-3}\frac{d}{dt}[t+1]$$

$$= -200(t+1)^{-3}$$

Evaluate at $t = 6$.

$$s'(6) = -200(7)^{-3}$$

$$= \frac{-200}{343} \implies \boxed{\approx -0.58 \text{ m/s}}$$

Extra Practice Questions

1. $f(x) = \sqrt[3]{5x-1}$

Rewrite:

$$f(x) = (5x-1)^{1/3}$$

Use the chain rule:

$$f'(x) = \frac{1}{3}(5x-1)^{-2/3} \cdot \frac{d}{dx}[5x-1]$$

$$= \frac{1}{3}(5x-1)^{-2/3}(5)$$

$$\boxed{f'(x) = \frac{5}{3}(5x-1)^{-2/3}}$$

$$\boxed{f'(x) = \frac{5}{3\sqrt[3]{(5x-1)^2}}}$$

2. $f(x) = 4^{2x}$

Use the exponential rule and the chain rule:

$$f'(x) = (\ln 4)\, 4^{2x} \cdot \frac{d}{dx}[2x]$$

$$= (\ln 4)\, 4^{2x}(2)$$

$$\boxed{f'(x) = 2(\ln 4)\, 4^{2x}}$$

3. $f(x) = x^2 e^{3x}$

Use the product rule:

$$f'(x) = \frac{d}{dx}[x^2] \cdot e^{3x} + x^2 \cdot \frac{d}{dx}[e^{3x}]$$

$$= 2x\, e^{3x} + x^2\left(e^{3x} \cdot 3\right)$$

$$= 2x\, e^{3x} + 3x^2 e^{3x}$$

Factor:

$$\boxed{f'(x) = e^{3x}(2x + 3x^2)}$$

$$\boxed{f'(x) = xe^{3x}(2 + 3x)}$$

4. $f(x) = \dfrac{1}{\sqrt{4x-2}}$

Rewrite:

$$f(x) = (4x-2)^{-1/2}$$

Use the chain rule:

$$f'(x) = -\frac{1}{2}(4x-2)^{-3/2} \cdot \frac{d}{dx}[4x-2]$$

$$= -\frac{1}{2}(4x-2)^{-3/2}(4)$$

$$\boxed{f'(x) = -2(4x-2)^{-3/2}}$$

$$\boxed{f'(x) = \frac{-2}{(4x-2)^{3/2}}}$$

6. $f(x) = \dfrac{(3-x)^7}{5x+4}$

Use the quotient rule:

$$f'(x) = \frac{\frac{d}{dx}[(3-x)^7](5x+4) - (3-x)^7 \frac{d}{dx}[5x+4]}{(5x+4)^2}$$

Compute each derivative:

$$\frac{d}{dx}[(3-x)^7] = 7(3-x)^6(-1) = -7(3-x)^6$$

$$\frac{d}{dx}[5x+4] = 5$$

Substitute:

$$f'(x) = \frac{-7(3-x)^6(5x+4) - 5(3-x)^7}{(5x+4)^2}$$

Factor $(3-x)^6$:

$$f'(x) = \frac{(3-x)^6[-7(5x+4) - 5(3-x)]}{(5x+4)^2}$$

Simplify inside the bracket:

$$-7(5x+4)-5(3-x) = -35x-28-15+5x = -30x-43$$

$$\boxed{f'(x) = \frac{(3-x)^6(-30x-43)}{(5x+4)^2}}$$

$$\boxed{f'(x) = -\frac{(3-x)^6(30x+43)}{(5x+4)^2}}$$

5. $f(x) = 3x^2\sqrt{x}$

Rewrite:

$$f(x) = 3x^2x^{1/2} = 3x^{5/2}$$

Use the power rule:

$$f'(x) = 3 \cdot \frac{5}{2}x^{3/2}$$

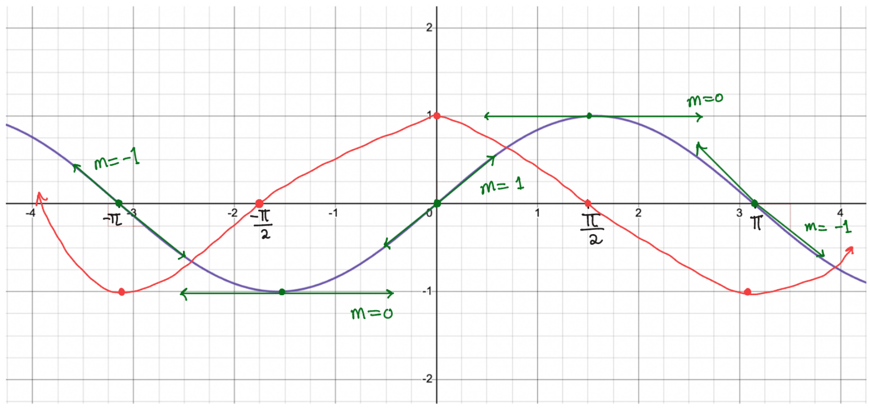
$$\boxed{f'(x) = \frac{15}{2}x^{3/2}}$$

$$\boxed{f'(x) = \frac{15}{2}x\sqrt{x}}$$

2.6 Derivatives of Trig Functions

Let's first look at

$$\frac{d}{dx}[\sin x]$$



From the graph and behavior of the function, we get:

$$\frac{d}{dx}[\sin x] = \cos x \quad \text{and} \quad \frac{d}{dx}[\cos x] = -\sin x$$

—

What about $\frac{d}{dx}[\tan x]$?
From our trig class, we recall that:

$$\tan x = \frac{\sin x}{\cos x}$$

Using the quotient rule:

$$\frac{d}{dx}[\tan x] = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \quad = \frac{1}{\cos^2 x} \quad \Longrightarrow \quad \boxed{\sec^2 x}$$

—

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\tan x] = \sec^2 x, \quad \frac{d}{dx}[\sec x] = \sec x \tan x,$$

$$\frac{d}{dx}[\cos x] = -\sin x, \quad \frac{d}{dx}[\cot x] = -\csc^2 x, \quad \frac{d}{dx}[\csc x] = -\csc x \cot x,$$

Examples

1. $f(x) = \sin(3x^2 - 5x + 1)$

We use the chain rule:

$$f'(x) = \cos(3x^2 - 5x + 1)\frac{d}{dx}[3x^2 - 5x + 1]$$

$$= \cos(3x^2 - 5x + 1)(6x - 5)$$

$$\boxed{f'(x) = \cos(3x^2 - 5x + 1)(6x - 5)}$$

2. $f(x) = \sec(1 - 4x)$

$$f'(x) = \sec(1 - 4x)\tan(1 - 4x)\frac{d}{dx}[1 - 4x]$$

$$= \sec(1 - 4x)\tan(1 - 4x)(-4)$$

$$\boxed{f'(x) = -4\sec(1 - 4x)\tan(1 - 4x)}$$

3. $f(x) = \cos^4 x$

Rewrite:

$$f(x) = (\cos x)^4$$

$$f'(x) = 4(\cos x)^3\frac{d}{dx}[\cos x]$$

$$= 4\cos^3 x(-\sin x)$$

$$\boxed{f'(x) = -4\cos^3 x \sin x}$$

4. $f(x) = \cot(2x + 3)$

$$f'(x) = -\csc^2(2x + 3)\frac{d}{dx}[2x + 3]$$

$$= -\csc^2(2x + 3)(2)$$

$$\boxed{f'(x) = -2\csc^2(2x + 3)}$$

5. $f(x) = \csc(5x)$

$$f'(x) = -\csc(5x)\cot(5x)\frac{d}{dx}[5x]$$

$$= -\csc(5x)\cot(5x)(5)$$

$$\boxed{f'(x) = -5\csc(5x)\cot(5x)}$$

6. $f(x) = \tan^2(6x + 3)$

Rewrite:

$$f(x) = [\tan(6x + 3)]^2$$

$$f'(x) = 2\tan(6x + 3)\frac{d}{dx}[\tan(6x + 3)]$$

$$= 2\tan(6x + 3)\sec^2(6x + 3)\frac{d}{dx}[6x + 3]$$

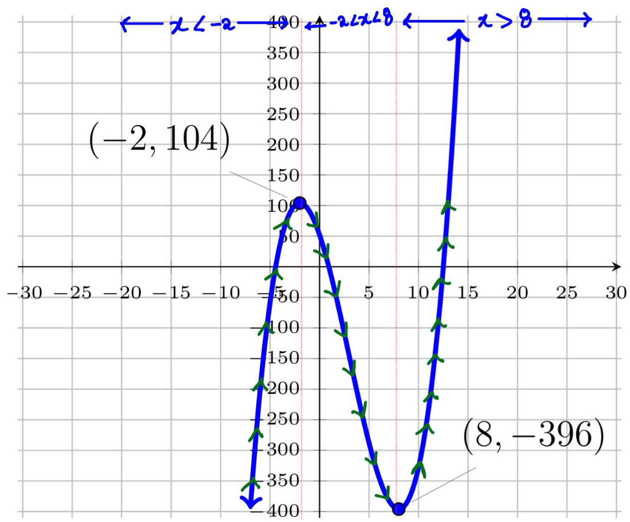
$$= 2\tan(6x + 3)\sec^2(6x + 3)(6)$$

$$\boxed{f'(x) = 12\tan(6x + 3)\sec^2(6x + 3)}$$

2.7 Applications of the First Derivative

Let us begin with the function

$$f(x) = x^3 - 9x^2 - 48x + 52.$$



What can we say about the derivative $f'(x)$?

- when $x < -2$? $\quad f'(x) > 0$
- when $x = -2$? $\quad f'(x) = 0$
- when $-2 < x < 8$? $\quad f'(x) < 0$
- when $x = 8$? $\quad f'(x) = 0$
- when $x > 8$? $\quad f'(x) > 0$

To justify this, we calculate the derivative and make a sign chart.

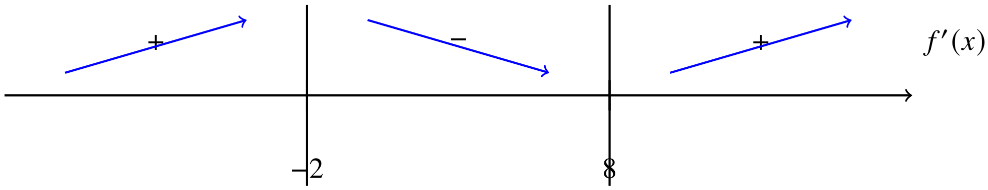
$$f'(x) = 3x^2 - 18x - 48 \quad \Rightarrow \quad 3(x^2 - 6x - 16) = 0 \quad \Rightarrow \quad 3(x - 8)(x + 2) = 0$$

So the critical points are

$$x = -2 \qquad \text{and} \qquad x = 8.$$

We test the sign of $f'(x)$ on the intervals determined by these critical points:

$$f'(-4) = 3(-12)(4) > 0, \qquad f'(0) = 3(-8)(2) < 0, \qquad f'(10) = 3(2)(12) > 0.$$



From the sign chart, we conclude:

$$f'(x) > 0 \text{ on } (-\infty, -2) \cup (8, \infty)$$

$$f'(x) < 0 \text{ on } (-2, 8)$$

So $f(x)$ is increasing on

$$(-\infty, -2) \cup (8, \infty)$$

and decreasing on

$$(-2, 8).$$

Also, since $f'(x)$ changes from positive to negative at $x = -2$, the function has a local maximum there. Since $f'(x)$ changes from negative to positive at $x = 8$, the function has a local minimum there.

Remark!
For a differentiable function $f(x)$:

- If $f'(x) > 0$, then the graph of $y = f(x)$ is increasing on that interval.
- If $f'(x) < 0$, then the graph of $y = f(x)$ is decreasing on that interval.

Furthermore, the graph of $y = f(x)$ can only have local maxima or minima at *critical points*, which are points where

$$f'(x) = 0 \qquad \text{or} \qquad f'(x) \text{ is undefined.}$$

Theorem: (First-Derivative Test for Local Extrema)

Suppose p is a critical point of a continuous function $f(x)$. Moving from left to right:

- If $f'(x)$ changes from negative to positive at p, then the graph of $y = f(x)$ has a local minimum at $x = p$.
- If $f'(x)$ changes from positive to negative at p, then the graph of $y = f(x)$ has a local maximum at $x = p$.

Example 1

Determine the intervals of increase/decrease and any local extrema for

$$f(x) = 2x^3 - 3x^2 - 36x + 8.$$

Step 1: Critical points.

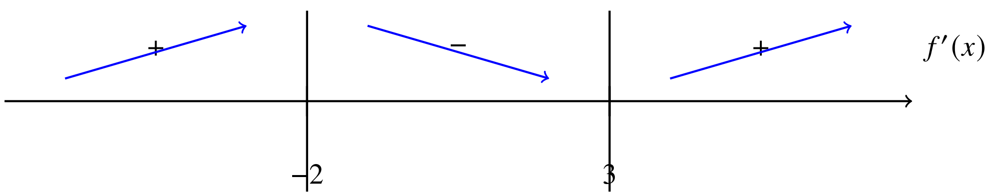
$$f'(x) = 6x^2 - 6x - 36 \quad\Rightarrow\quad 6(x^2 - x - 6) = 0 \quad\Rightarrow\quad 6(x-3)(x+2) = 0$$

So the critical points are

$$x = -2 \qquad \text{and} \qquad x = 3.$$

Step 2: Sign chart for $f'(x)$.
We test values in each interval:

$$f'(-3) = 6(-6)(-1) > 0, \qquad f'(0) = 6(-3)(2) < 0, \qquad f'(4) = 6(1)(6) > 0.$$



Therefore,

$$\text{Increasing: } (-\infty, -2) \cup (3, \infty)$$

$$\text{Decreasing: } (-2, 3)$$

Since $f'(x)$ changes from positive to negative at $x = -2$, there is a local maximum at $x = -2$.
Since $f'(x)$ changes from negative to positive at $x = 3$, there is a local minimum at $x = 3$.

Example 2

Determine the intervals of increase/decrease and any local extrema for

$$f(x) = x^5 - 10x^3 - 8.$$

Step 1: Critical points.

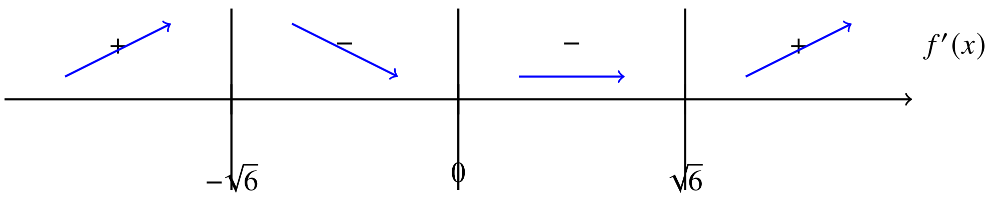
$$f'(x) = 5x^4 - 30x^2 \quad\Rightarrow\quad 5x^2(x^2 - 6) = 0$$

So the critical points are

$$x = 0, \qquad x = \pm\sqrt{6}.$$

Step 2: Sign chart for $f'(x)$.
We test values in each interval:

$$f'(-3) = 5(9)(3) > 0, \quad f'(-1) = 5(1)(-5) < 0, \quad f'(1) = 5(1)(-5) < 0, \quad f'(3) = 5(9)(3) > 0.$$



Therefore,

$$\text{Increasing: } (-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty)$$

$$\text{Decreasing: } (-\sqrt{6}, 0) \cup (0, \sqrt{6})$$

Since $f'(x)$ changes from positive to negative at $x = -\sqrt{6}$, there is a local maximum at $x = -\sqrt{6}$.
Since $f'(x)$ changes from negative to positive at $x = \sqrt{6}$, there is a local minimum at $x = \sqrt{6}$.
At $x = 0$, the sign of $f'(x)$ does not change, so there is no local extremum there.

Example 3

An article in a sociology journal stated that if a particular health-care program for the elderly were initiated, then t years after its start, n thousand elderly people would receive direct benefits, where

$$n = \frac{t^3}{3} - 6t^2 + 32t, \qquad 0 \le t \le 12.$$

Determine when the maximum number of people will receive the benefits.
Step 1: Critical points.

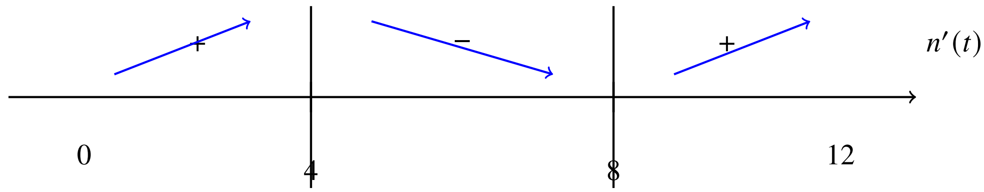
$$n'(t) = t^2 - 12t + 32 \quad \Rightarrow \quad (t-8)(t-4) = 0$$

So the critical points are

$$t = 4 \qquad \text{and} \qquad t = 8.$$

Step 2: Sign chart for $n'(t)$.
We test values in the intervals:

$$n'(0) = 32 > 0, \qquad n'(5) = (-3)(1) < 0, \qquad n'(9) = (1)(5) > 0.$$



This tells us that $n(t)$ is increasing on $(0,4)$, decreasing on $(4,8)$, and increasing on $(8,12)$. So there is a local maximum at $t = 4$ and a local minimum at $t = 8$.

Step 3: To check the critical points and the endpoints.

$$n(0) = \frac{0^3}{3} - 6(0)^2 + 32(0) = 0$$

$$n(4) = \frac{4^3}{3} - 6(4)^2 + 32(4) = \frac{64}{3} - 96 + 128 = \frac{160}{3} \approx 53.33$$

$$n(8) = \frac{8^3}{3} - 6(8)^2 + 32(8) = \frac{512}{3} - 384 + 256 = -\frac{128}{3} \approx -42.67$$

$$n(12) = \frac{12^3}{3} - 6(12)^2 + 32(12) = 576 - 864 + 384 = 96$$

Therefore, the maximum value occurs at

$$t = 12.$$

So the maximum number of people will receive benefits

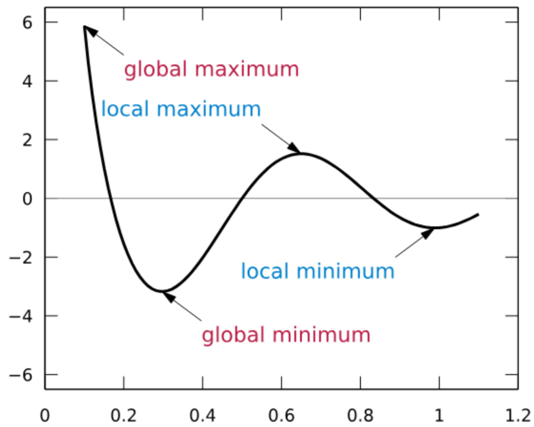
$$\boxed{\text{12 years after the program starts.}}$$

Since n is measured in thousands, the maximum number of people is

$$\boxed{\text{96,000 people.}}$$

2.8 Global Extrema

We have seen local extrema:



In this graph, we can identify both local and global extrema.
The highest point on the entire graph is called the *global maximum*, while the lowest point on the entire graph is called the *global minimum*.
We can also observe smaller peaks and valleys within the graph. These are called *local extrema*:

- A *local maximum* is a point where the function is greater than nearby values.
- A *local minimum* is a point where the function is less than nearby values.

Theorem: (Extreme Value Theorem)
If f is continuous on the closed interval $[a, b]$, then f has a global maximum and a global minimum on that interval.

Example 1

Determine the global extrema for each function on the given interval.

(a) $f(x) = 3x - 4\sin x, \qquad [0, \pi]$
Step 1: Critical points on $[0, \pi]$**.**

$$f'(x) = 3 - 4\cos x \quad\Rightarrow\quad 3 = 4\cos x \quad\Rightarrow\quad \frac{3}{4} = \cos x \quad\Rightarrow\quad x = \arccos\left(\frac{3}{4}\right)$$

Step 2: Plug in the critical points and the endpoints.

$$f(0) = 0$$

$$f\left(\arccos\left(\frac{3}{4}\right)\right) \approx -0.48$$

$$f(\pi) = 3\pi \approx 9.42$$

Therefore, the global minimum is approximately $\boxed{-0.48}$ and the global maximum is $\boxed{3\pi \approx 9.42}$.

(b) $f(x) = xe^{-x^2/2}, \qquad [-2, 2]$

Step 1: Find the critical points on $[-2, 2]$**.**

$$f'(x) = (1)e^{-x^2/2} + x\, e^{-x^2/2}(-x)$$

$$= e^{-x^2/2} - x^2 e^{-x^2/2}$$

$$= e^{-x^2/2}(1 - x^2)$$

Set $f'(x) = 0$:

$$e^{-x^2/2}(1 - x^2) = 0$$

Since

$$e^{-x^2/2} \neq 0,$$

we must have

$$1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1$$

Step 2: Plug in the critical points and the endpoints.

$$f(-2) = (-2)e^{-(-2)^2/2} = -2e^{-2} \approx -0.27$$

$$f(-1) = (-1)e^{-(-1)^2/2} = -e^{-1/2} \approx -0.61$$

$$f(1) = (1)e^{-(1)^2/2} = e^{-1/2} \approx 0.61$$

$$f(2) = (2)e^{-(2)^2/2} = 2e^{-2} \approx 0.27$$

Therefore, the global minimum is

$$\boxed{-e^{-1/2} \approx -0.61}$$

at $x = -1$, and the global maximum is

$$\boxed{e^{-1/2} \approx 0.61}$$

at $x = 1$.

Example 2

A company has $100,000 to spend for equipment and labor combined. The company spends x thousand dollars on equipment and $100 - x$ on labor, enabling it to produce Q items, where

$$Q = 5x^{0.3}(100 - x)^{0.8}, \qquad 0 \le x \le 100$$

How much should the company spend on equipment to maximize production? On labor? What is the maximum production?

Step 1: Find the critical points on $[0, 100]$**.**

$$\begin{aligned} Q' &= 5(0.3)x^{-0.7}(100 - x)^{0.8} + 5x^{0.3}(0.8)(100 - x)^{-0.2}(-1) \\ &= 1.5x^{-0.7}(100 - x)^{0.8} - 4x^{0.3}(100 - x)^{-0.2} \end{aligned}$$

Set $Q' = 0$:

$$1.5x^{-0.7}(100 - x)^{0.8} - 4x^{0.3}(100 - x)^{-0.2} = 0$$

Move one term to the other side:

$$1.5x^{-0.7}(100 - x)^{0.8} = 4x^{0.3}(100 - x)^{-0.2}$$

Multiply both sides by $x^{0.7}(100 - x)^{0.2}$:

$$1.5(100 - x) = 4x$$

$$150 - 1.5x = 4x \quad \Rightarrow \quad 150 = 5.5x \quad \Rightarrow \quad x \approx 27.27$$

Step 2: Plug in the critical point and the endpoints.

$$Q(0) = 0$$

$$Q(27.27) \approx 5(27.27)^{0.3}(100 - 27.27)^{0.8} \approx 415.96$$

$$Q(100) = 0$$

Therefore, the maximum production is approximately

$$\boxed{\text{416 items}}.$$

So the company should spend

$$\boxed{\text{\$27,273 on equipment}}$$

and

$$\boxed{\text{\$72,727 on labor}}.$$

Chapter 3: Multiple Derivatives, concavity, linear approximations, implicit differentiation, and limits with L'Hôpital's Rule

3.1 Multiple Derivatives

So far we have taken the *first derivative* of a function. But we can keep going: we can take the derivative of the derivative, and so on. These are called **higher-order derivatives**.

Notation

Higher-Order Derivative Notation:

Order	**Notation**	**Meaning**
First	$f'(x)$	Derivative of $f(x)$
Second	$f''(x) = \frac{d^2}{dx^2}[f(x)]$	Derivative of $f'(x)$
Third	$f'''(x) = \frac{d^3}{dx^3}[f(x)]$	Derivative of $f''(x)$
Fourth	$f^{(4)}(x) = \frac{d^4}{dx^4}[f(x)]$	Derivative of $f'''(x)$
n-th	$f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]$	Derivative taken n times

Note: Starting from the fourth derivative, we stop using prime notation and switch to a number in parentheses to avoid confusion with powers.

Physical Meaning of the Second Derivative: If $s(t)$ is the **position** of an object at time t, then:

$$s'(t) = v(t) \quad \text{(velocity)} \qquad \text{and} \qquad s''(t) = v'(t) = a(t) \quad \text{(acceleration).}$$

To determine whether a particle is **speeding up** or **slowing down** at a given moment, compare the signs of $v(t)$ and $a(t)$:

Sign of $v(t)$	Sign of $a(t)$	Conclusion
+	+	Speeding up
−	−	Speeding up
+	−	Slowing down
−	+	Slowing down

Rule: Same signs ⇒ speeding up. Opposite signs ⇒ slowing down.

Example 3.1: Calculate the second derivative of each function

(a) $f(x) = 4x^5 - 3x^2 + 7x + 8$

Step 1: Find the first derivative.

$$f'(x) = 20x^4 - 6x + 7$$

Step 2: Differentiate again to get the second derivative.

$$f''(x) = \frac{d}{dx}\left[20x^4 - 6x + 7\right]$$

$$\boxed{f''(x) = 80x^3 - 6}$$

(b) $f(x) = \sin(2x)$

Step 1: Find the first derivative.

Using the chain rule with inner function $u = 2x$:

$$f'(x) = \cos(2x) \cdot \frac{d}{dx}[2x] = 2\cos(2x)$$

Step 2: Differentiate again.

Again using the chain rule:

$$f''(x) = \frac{d}{dx}[2\cos(2x)] = 2 \cdot (-\sin(2x)) \cdot \frac{d}{dx}[2x]$$

$$= 2 \cdot (-\sin(2x)) \cdot 2 \implies \boxed{f''(x) = -4\sin(2x)}$$

Example 3.2

For $f(x) = \sqrt{x}$, compute $f'''(x)$

Rewrite the square root using a fractional exponent:

$$f(x) = x^{1/2}$$

Step 1: First derivative.

$$f'(x) = \frac{1}{2}x^{-1/2}$$

Step 2: Second derivative.

$$f''(x) = \frac{d}{dx}\left[\frac{1}{2}x^{-1/2}\right] = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-3/2} = -\frac{1}{4}x^{-3/2}$$

Step 3: Third derivative.

$$f'''(x) = \frac{d}{dx}\left[-\frac{1}{4}x^{-3/2}\right] = -\frac{1}{4} \cdot \left(-\frac{3}{2}\right) x^{-5/2} = \frac{3}{8}x^{-5/2}$$

$$\boxed{f'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8\,x^{5/2}}}$$

Example 3.3: Position, velocity, and acceleration

Let $s(t) = t^3 - 12t^2 + 45t$ be the position of a particle (in ft) after t seconds.

1. What is the velocity at $t = 1$ second?
2. What is the acceleration at $t = 1$ second?
3. Is the particle slowing down or speeding up at $t = 1$ second?

Step 1: Find velocity and acceleration.

$$v(t) = s'(t) = 3t^2 - 24t + 45$$

$$a(t) = s''(t) = v'(t) = 6t - 24$$

Step 2: Evaluate at $t = 1$.

$$v(1) = 3(1)^2 - 24(1) + 45 = 3 - 24 + 45 = \boxed{24 \text{ ft/s}}$$

$$a(1) = 6(1) - 24 = \boxed{-18 \text{ ft/s}^2}$$

Step 3: Speeding up or slowing down?

At $t = 1$: $v(1) = 24 > 0$ and $a(1) = -18 < 0$. Since v and a have *opposite* signs, the particle is

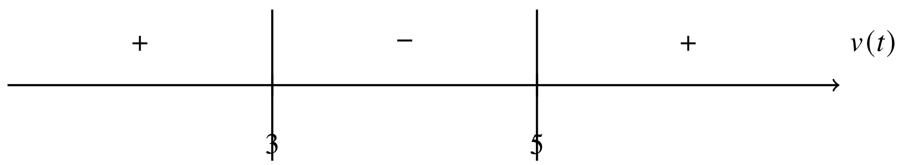
$$\boxed{\text{slowing down at } t = 1.}$$

Complete analysis using sign charts:

Sign chart for $v(t) = 3t^2 - 24t + 45$*:*
Set $v(t) = 0$:

$$3(t^2 - 8t + 15) = 0 \implies 3(t-3)(t-5) = 0 \implies t = 3, \quad t = 5.$$

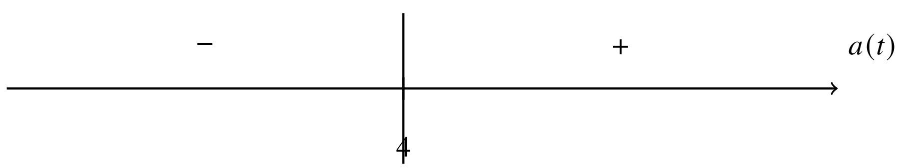
Test values: $v(0) = 45 > 0$, $v(4) = 3(1)(-1) = -3 < 0$, $v(6) = 3(3)(1) = 9 > 0$.



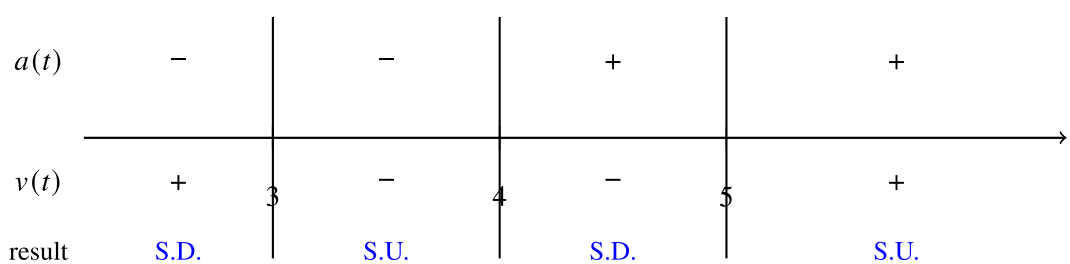
Sign chart for $a(t) = 6t - 24$*:*
Set $a(t) = 0$:

$$6t - 24 = 0 \implies t = 4.$$

Test values: $a(0) = -24 < 0$, $a(5) = 6 > 0$.



Combined sign chart (speeding up / slowing down):



Therefore:

- **Slowing down** on $(0, 3)$ and $(4, 5)$
- **Speeding up** on $(3, 4)$ and $(5, \infty)$

Example 3.4: Speeding up or slowing down at a given position

A particle moves along a line so that its position at any time t is given by

$$f(t) = 2t^2 - 3t - 3,$$

where f is measured in meters and t in seconds. Is the particle speeding up or slowing down when the position is 6 meters?

Step 1: We first find the time t when the position equals 6 meters.
Set $f(t) = 6$:

$$2t^2 - 3t - 3 = 6$$

$$2t^2 - 3t - 9 = 0$$

Factor:

$$(2t + 3)(t - 3) = 0$$

$$t = -\frac{3}{2} \quad \text{or} \quad t = 3.$$

Since time must be non-negative, we use

$$t = 3 \text{ seconds}.$$

Step 2: Find velocity and acceleration.

$$v(t) = f'(t) = 4t - 3$$

$$a(t) = f''(t) = 4$$

Step 3: Evaluate at $t = 3$.

$$v(3) = 4(3) - 3 = 12 - 3 = 9 \text{ m/s}$$

$$a(3) = 4 \text{ m/s}^2$$

Step 4: Compare signs.
$v(3) = 9 > 0$ and $a(3) = 4 > 0$. Both are *positive* (same sign).

$$\boxed{\text{The particle is speeding up when its position is 6 meters.}}$$

3.2 Concavity

Concavity describes *how* a function bends. Even two increasing functions can bend differently — one cups upward like a bowl, the other arches downward like a hill. The second derivative tells us which is which.

Definition (Concave Up and Concave Down):
A function $f(x)$ is called **concave up** on an interval (a, b) if its graph bends upward like a cup $\cup$. Tangent lines lie *below* the graph.

$$f''(x) > 0 \quad \text{on } (a, b) \implies \text{concave up}$$

A function $f(x)$ is called **concave down** on an interval (a, b) if its graph bends downward like a cap $\cap$. Tangent lines lie *above* the graph.

$$f''(x) < 0 \quad \text{on } (a, b) \implies \text{concave down}$$

Shape	**Sign of** $f''(x)$	**Description**
$\cup$ (cup)	$f''(x) > 0$	Concave up
$\cap$ (cap)	$f''(x) < 0$	Concave down

Definition (Inflection Point): A point p is a **possible inflection point** of $f(x)$ if

$$f''(p) = 0 \qquad \text{or} \qquad f''(p) \text{ is undefined.}$$

To confirm that p is an *actual* inflection point, check that f'' **changes sign** at p (using a sign chart). If the sign does not change, there is no inflection point at p.

Example 3.7

Find the inflection points for $f(x) = \frac{1}{4}x^4 - 4x^3 + 24x^2$

Step 1: Find the second derivative.

$$f'(x) = x^3 - 12x^2 + 48x$$

$$f''(x) = 3x^2 - 24x + 48 = 3(x^2 - 8x + 16) = 3(x-4)^2$$

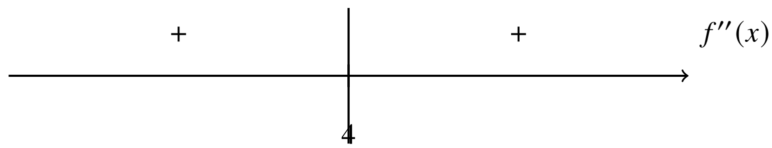
Step 2: Set $f''(x) = 0$ to find possible inflection points.

$$3(x-4)^2 = 0 \implies x = 4$$

Step 3: Check whether f'' changes sign at $x = 4$.
Test values on each side:

$$f''(3) = 3(3-4)^2 = 3(1) = 3 > 0$$

$$f''(5) = 3(5-4)^2 = 3(1) = 3 > 0$$



Since $f''(x)$ does *not* change sign at $x = 4$, there is **no inflection point**.

$$\boxed{\text{The function has no inflection points. It is concave up everywhere.}}$$

Example 3.8

Find the inflection points for $f(x) = xe^{-x}$

Step 1: Find the first derivative.
Using the product rule:

$$f'(x) = (1)e^{-x} + x \cdot (-e^{-x}) = e^{-x} - xe^{-x} = e^{-x}(1-x)$$

Step 2: Find the second derivative.
Differentiate $f'(x) = e^{-x}(1-x)$ using the product rule:

$$f''(x) = (-e^{-x})(1-x) + e^{-x}(-1)$$

$$= -e^{-x}(1-x) - e^{-x}$$

$$= -e^{-x}\left[(1-x) + 1\right]$$

$$= \boxed{-e^{-x}(2-x)} \quad = \quad e^{-x}(x-2)$$

Step 3: Set $f''(x) = 0$ to find possible inflection points.

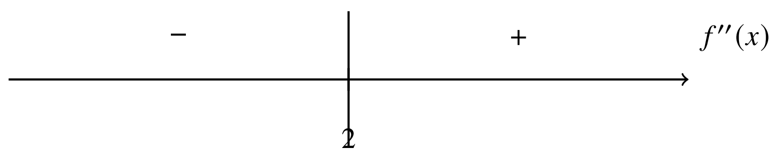
Since $e^{-x} \neq 0$ for all x, we need:

$$x - 2 = 0 \implies x = 2$$

Step 4: Check whether f'' changes sign at $x = 2$.
Test values:

$$f''(0) = e^{0}(0 - 2) = -2 < 0$$

$$f''(3) = e^{-3}(3 - 2) = e^{-3} > 0$$



f'' changes from negative to positive at $x = 2$, so there is an inflection point.

The y-value at the inflection point:

$$f(2) = 2e^{-2}$$

$$\boxed{\text{Inflection point at } \left(2,\ 2e^{-2}\right).}$$

The graph is **concave down** on $(-\infty, 2)$ and **concave up** on $(2, \infty)$.

Theorem 3.9: (Second-Derivative Test for Local Extrema)

Let $f(x)$ be a differentiable function and let p be a critical point where $f'(p) = 0$.

Condition	**Sign of $f''(p)$**	**Conclusion**
$f'(p) = 0$ and $f''(p) > 0$	+	Local **minimum** at p $\cup$
$f'(p) = 0$ and $f''(p) < 0$	−	Local **maximum** at p $\cap$
$f'(p) = 0$ and $f''(p) = 0$	0	**Inconclusive** — use first-derivative test

Example 3.10

Determine the local extrema for $f(x) = \dfrac{x}{x^2+1}$

Step 1: Find critical points.

Use the quotient rule:

$$f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Set $f'(x) = 0$:

$$\frac{1-x^2}{(x^2+1)^2} = 0 \implies 1 - x^2 = 0 \implies x = \pm 1$$

Can $f'(x)$ be undefined? We need $(x^2+1)^2 = 0$, which gives $x^2 = -1$ — no real solution.

So the critical points are $x = 1$ and $x = -1$.

Step 2: Apply the second-derivative test.

Find $f''(x)$ by differentiating $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$ using the quotient rule:

$$f''(x) = \frac{(-2x)(x^2+1)^2 - (1-x^2)\cdot 2(x^2+1)(2x)}{(x^2+1)^4}$$

Factor (x^2+1) from the numerator:

$$= \frac{(x^2+1)\left[(-2x)(x^2+1) - 4x(1-x^2)\right]}{(x^2+1)^4}$$

$$= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3}$$

$$= \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3} = \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Evaluate at the critical points:

At $x = 1$:

$$f''(1) = \frac{2(1)(1-3)}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2} < 0 \implies \text{local } \textbf{maximum} \text{ at } x = 1$$

At $x = -1$:

$$f''(-1) = \frac{2(-1)(1-3)}{(1+1)^3} = \frac{4}{8} = \frac{1}{2} > 0 \implies \text{local } \textbf{minimum} \text{ at } x = -1$$

$$\boxed{\text{Local maximum at } x = 1 \qquad \text{Local minimum at } x = -1}$$

3.3 Sorting Out the Tests for Extrema

We now have *two* tests for finding local extrema. Here is how to choose between them.

First-Derivative Test vs. Second-Derivative Test

	First-Derivative Test	**Second-Derivative Test**
How it works	Build a sign chart for $f'(x)$ and check if the sign changes at a critical point	Evaluate $f''(p)$ at a critical point p
Always gives an answer?	**Yes** — always determines max, min, or neither	**No** — inconclusive when $f''(p) = 0$
Best when	f'' is hard to compute, or $f''(p) = 0$	f'' is easy to compute

Key rule: If $f''(p) = 0$, the second-derivative test fails. You must go back and use the first-derivative test.

Example 3.11

Determine the local extrema for each function.

(a) $f(x) = (x^2 - 4)^{2/3}$
Step 1: Find the critical points.

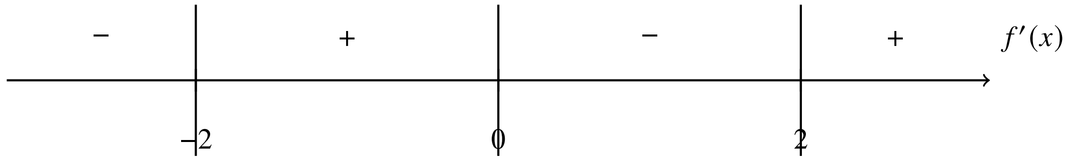
$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3} \cdot (2x) = \frac{4x}{3(x^2 - 4)^{1/3}}$$

$f'(x) = 0$: numerator $= 0 \implies 4x = 0 \implies x = 0$.
$f'(x)$ undefined: denominator $= 0 \implies x^2 - 4 = 0 \implies x = \pm 2$.
Critical points: $x = -2,\ 0,\ 2$.

Step 2: Since f'' is complicated, use the first-derivative test.
Test values in each interval:

$$f'(-3) = \frac{-12}{3\sqrt[3]{5}} < 0 \qquad f'(-1) = \frac{-4}{3\sqrt[3]{-3}} = \frac{-4}{-3\sqrt[3]{3}} > 0$$

$$f'(1) = \frac{4}{3\sqrt[3]{-3}} < 0 \qquad f'(3) = \frac{12}{3\sqrt[3]{5}} > 0$$



- At $x = -2$: f' changes from $-$ to $+$ $\implies$ **local minimum**
- At $x = 0$: f' changes from $+$ to $-$ $\implies$ **local maximum**
- At $x = 2$: f' changes from $-$ to $+$ $\implies$ **local minimum**

(b) $g(x) = x^3 - 4x^2 + 10$

Step 1: Find the critical points.

$$g'(x) = 3x^2 - 8x = x(3x - 8) = 0 \implies x = 0 \quad \text{or} \quad x = \frac{8}{3}$$

Step 2: Use the second-derivative test (since g'' is easy).

$$g''(x) = 6x - 8$$

At $x = 0$:

$$g''(0) = -8 < 0 \implies \textbf{local maximum at } x = 0$$

At $x = \frac{8}{3}$:

$$g''\left(\frac{8}{3}\right) = 6 \cdot \frac{8}{3} - 8 = 16 - 8 = 8 > 0 \implies \textbf{local minimum at } x = \frac{8}{3}$$

$$\boxed{\text{Local maximum at } x = 0 \qquad \text{Local minimum at } x = \frac{8}{3}}$$

3.4 Linear Approximations and Calculating Error

Since the tangent line at $x = a$ lies very close to the graph of $f(x)$ when x is close to a, we can use it to *approximate* the value of the function near a.

Theorem 3.14: **Linear Approximation**

Let $f(x)$ be a differentiable function at $x = a$. Then for x near a:

$$f(x) \approx f(a) + f'(a)(x - a)$$

This is the equation of the tangent line to $y = f(x)$ at $x = a$.

How to use it:

Find the tangent line at a nearby known point a, then plug in the value you want to estimate.

$$\underbrace{a}_{\text{known point}} \xrightarrow{\text{tangent line } y=f(a)+f'(a)(x-a)} \underbrace{x}_{\text{estimate here}}$$

Example 3.15

Use a tangent line approximation to estimate $\sqrt[3]{9}$.

Step 1: Pick the function and a nearby known value.

$$f(x) = x^{1/3}, \qquad a = 8 \quad \text{since } \sqrt[3]{8} = 2 \text{ exactly, and 8 is close to 9.}$$

Step 2: Find the point.

$$f(8) = 2 \implies \text{Point: } (8,\ 2)$$

Step 3: Find the slope.

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} \qquad \implies \qquad m = f'(8) = \frac{1}{3(2)^2} = \frac{1}{12}$$

Step 4: Write the tangent line.

$$2 = \frac{1}{12}(8) + b \implies b = \frac{4}{3} \qquad \implies \qquad y = \frac{1}{12}x + \frac{4}{3}$$

Step 5: Estimate.

$$\sqrt[3]{9} \approx \frac{1}{12}(9) + \frac{4}{3} = \frac{9}{12} + \frac{16}{12} = \frac{25}{12} \approx \boxed{2.083}$$

Example 3.16

Use a tangent line approximation to estimate $\sin(0.2)$.

We choose $f(x) = \sin x$ and $a = 0$ since $\sin(0) = 0$ exactly, and 0 is close to 0.2.
Point: $f(0) = 0 \implies (0,\ 0)$
Slope: $f'(x) = \cos x \implies m = f'(0) = \cos(0) = 1$
Tangent line: $y = x$

Near $x = 0$: $\sin x \approx x$, so

$$\sin(0.2) \approx \boxed{0.2}$$

Example 3.17

A particle's position at $t = 1$ second is 7 meters and its velocity at $t = 1$ second is 800 m/s. Estimate the position at $t = 1.15$ seconds.

We are given: $\Delta(1) = 7$ and $\Delta'(1) = 800$.
Point: $(1,\ 7)$ **Slope:** $m = 800$
Tangent line:

$$7 = 800(1) + b \implies b = -793 \implies \Delta(t) \approx 800t - 793$$

Estimate:

$$\Delta(1.15) \approx 800(1.15) - 793 = 920 - 793 = \boxed{127 \text{ m}}$$

Example 3.18

Use a tangent line approximation to estimate $\sqrt{8}$.

We choose $f(x) = \sqrt{x}$ and $a = 9$ since $\sqrt{9} = 3$ exactly, and 9 is close to 8.

Point: $f(9) = 3 \implies (9,\ 3)$
Slope: $f'(x) = \dfrac{1}{2\sqrt{x}} \implies m = f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$
Tangent line: $y = 3 + \dfrac{1}{6}(x - 9)$
Estimate:

$$\sqrt{8} \approx 3 + \frac{1}{6}(8 - 9) = 3 - \frac{1}{6} = \frac{17}{6} \approx \boxed{2.833}$$

Differentials

We know that $f'(x)$ measures how fast y changes relative to x. When x changes by a small amount, we can estimate how much y changes.

We call dx the **differential of** x and dy the **differential of** y:

$$dy = f'(x)\,dx$$

Think of it this way:

- dx is the small change in x (same as Δx)
- dy is the **estimated** change in y (approximation of Δy)
- Δy is the **actual** change in $y = f(x + dx) - f(x)$

The differential dy gives us a quick estimate of how much the output changes when the input changes by dx – without computing the function twice.

Example 3.20

Consider $f(x) = x^2 + 2x$. Compute both the actual change Δy and the estimated change dy if the change in x is 0.1 at $x = 0$ and at $x = 3$.

First find the differential:

$$f'(x) = 2x + 2 \implies dy = (2x + 2)\,dx$$

At $x = 0$**:**

$$\Delta y = f(0.1) - f(0) = \left[(0.1)^2 + 2(0.1)\right] - 0 = 0.01 + 0.2 = 0.21$$

$$dy = (2(0) + 2)(0.1) = 2(0.1) = 0.2$$

At $x = 3$**:**

$$\Delta y = f(3.1) - f(3) = \left[(3.1)^2 + 2(3.1)\right] - \left[9 + 6\right] = 15.81 - 15 = 0.81$$

$$dy = (2(3) + 2)(0.1) = 8(0.1) = 0.8$$

Example 3.21

Consider $f(x) = \sqrt{x}$. Compute both the actual change Δy and the estimated change dy if the change in x is 1 at $x = 1$ and at $x = 9$.

First find the differential:

$$f'(x) = \frac{1}{2\sqrt{x}} \implies dy = \frac{1}{2\sqrt{x}}\,dx$$

At $x = 1$:

$$\Delta y = f(2) - f(1) = \sqrt{2} - 1 \approx 0.414$$

$$dy = \frac{1}{2\sqrt{1}}(1) = \frac{1}{2} = 0.5$$

At $x = 9$:

$$\Delta y = f(10) - f(9) = \sqrt{10} - 3 \approx 0.162$$

$$dy = \frac{1}{2\sqrt{9}}(1) = \frac{1}{6} \approx 0.167$$

Example 3.22

Use a linear approximation to estimate $\ln 2$. Will it be an overestimate or an underestimate?

We choose $f(x) = \ln x$ and $a = 1$ since $\ln 1 = 0$ exactly, and 1 is close to 2.

Point: $f(1) = 0 \implies (1,\ 0)$
Slope: $f'(x) = \frac{1}{x} \implies m = f'(1) = 1$
Tangent line:

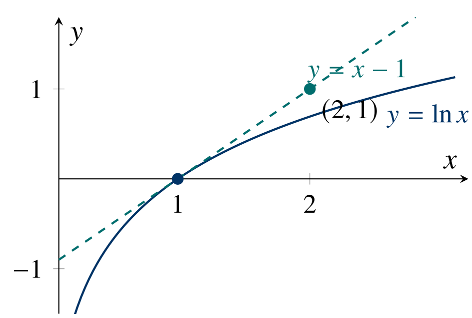
$$0 = 1(1) + b \implies b = -1 \implies y = x - 1$$

Estimate:

$$\ln 2 \approx 2 - 1 = \boxed{1}$$

Overestimate or underestimate?
$f''(x) = -\frac{1}{x^2} < 0$, so $\ln x$ is **concave down** everywhere. When a graph is concave down, the tangent line lies *above* the graph.

$$\boxed{\text{This is an overestimate.}}$$



Example 3.23

Use a linear approximation to estimate $\tan\frac{\pi}{7}$. Will it be an overestimate or an underestimate?

We choose $f(x) = \tan x$ and $x = \frac{\pi}{6}$ since $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ exactly, and $\frac{\pi}{6}$ is close to $\frac{\pi}{7}$.

Point: $f\left(\frac{\pi}{6}\right) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \implies \left(\frac{\pi}{6}, \frac{1}{\sqrt{3}}\right)$

Slope: $f'(x) = \sec^2 x \implies m = \sec^2\left(\frac{\pi}{6}\right) = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$

Tangent line:

$$\frac{1}{\sqrt{3}} = \frac{4}{3}\cdot\frac{\pi}{6} + b \implies b = \frac{1}{\sqrt{3}} - \frac{2\pi}{9} \implies y = \frac{4}{3}x + \frac{1}{\sqrt{3}} - \frac{2\pi}{9}$$

Estimate:

$$\tan\frac{\pi}{7} \approx \frac{4}{3}\cdot\frac{\pi}{7} + \frac{1}{\sqrt{3}} - \frac{2\pi}{9} \approx \boxed{0.478}$$

Overestimate or underestimate?

$f''(x) = 2\sec^2 x\tan x > 0$ near $x = \frac{\pi}{6}$, so $\tan x$ is **concave up** there. When a graph is concave up, the tangent line lies *below* the graph.

$$\boxed{\text{This is an underestimate.}}$$

Example 3.24

The sides of a cube are found to be 2 ft with a possible error of no more than 1.25 inches. What is the maximum possible error in the volume?

First convert the error to feet:

$$dx = 1.25 \text{ in} \times \frac{1 \text{ ft}}{12 \text{ in}} \approx 0.104 \text{ ft}$$

Volume of a cube: $V = x^3$

$$\frac{dV}{dx} = 3x^2 \implies dV = 3x^2\,dx$$

$$dV = 3(2)^2(0.104) = 3(4)(0.104) = \boxed{1.248 \text{ ft}^3}$$

3.5 Implicit Differentiation and Related Rates

So far we have differentiated functions written explicitly as $y = f(x)$. But some equations like $x^2 + y^2 = 9$ cannot be easily solved for y. We can still find $\frac{dy}{dx}$ using **implicit differentiation**.

Key Idea!

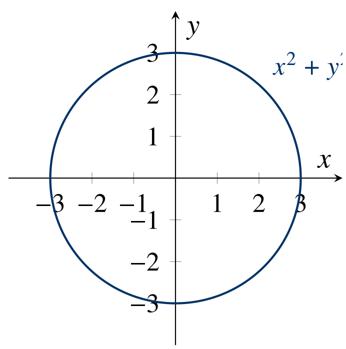
When differentiating both sides of an equation with respect to x:

- Terms with only x — differentiate normally.
- Terms with y — differentiate normally but multiply by $\frac{dy}{dx}$ (chain rule, since y depends on x).

In short:

$$\frac{d}{dx}\big[f(y)\big] = \frac{d}{dy}\big[f(y)\big] \cdot \frac{dy}{dx}$$

Consider the circle $x^2 + y^2 = 9$ centered at the origin with radius 3. This graph is *not* the graph of a function, but drawing tangent lines still makes sense. We can find the slope of the tangent line at any point using implicit differentiation.



$$\frac{d}{dx}\big[x^2 + y^2\big] = \frac{d}{dx}\big[9\big]$$

$$2x + \frac{d}{dx}\big[y^2\big] = 0$$

The term $\frac{d}{dx}[y^2]$ does *not* match our derivative rules directly since it is in y, not x. We apply the chain rule:

$$\frac{d}{dx}\big[y^2\big] = \frac{d}{dy}\big[y^2\big] \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$$

So we get:

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies 2y \cdot \frac{dy}{dx} = -2x \implies \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Example 3.26

Calculate $\frac{dy}{dx}$ (the derivative) implicitly for each function.

(a) $3y^2 + x^4 = 10$

First, differentiate both sides with respect to x:

$$\frac{d}{dx}\left[3y^2 + x^4\right] = \frac{d}{dx}[10]$$

$$\frac{d}{dx}\left[3y^2\right] + \frac{d}{dx}\left[x^4\right] = 0$$

The x^4 term differentiates normally. For $3y^2$, we apply the chain rule since y depends on x:

$$\frac{d}{dy}\left[3y^2\right] \cdot \frac{dy}{dx} + 4x^3 = 0$$

$$6y \cdot \frac{dy}{dx} + 4x^3 = 0$$

Isolate $\frac{dy}{dx}$:

$$6y \cdot \frac{dy}{dx} = -4x^3 \implies \boxed{\frac{dy}{dx} = \frac{-4x^3}{6y} = \frac{-2x^3}{3y}}$$

(b) $x^2 + y^2 = 2xy$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = \frac{d}{dx}[2xy]$$

The right side requires the product rule:

$$2x + 2y \cdot \frac{dy}{dx} = 2 \cdot y + 2x \cdot \frac{dy}{dx}$$

Collect all $\frac{dy}{dx}$ terms on the left:

$$2y \cdot \frac{dy}{dx} - 2x \cdot \frac{dy}{dx} = 2y - 2x \implies (2y - 2x)\frac{dy}{dx} = 2y - 2x$$

Divide both sides by $(2y - 2x)$:

$$\boxed{\frac{dy}{dx} = 1}$$

(c) $\sin x + \cos y = 5x - 4y$

Differentiate both sides with respect to x:

$$\frac{d}{dx}[\sin x] + \frac{d}{dx}[\cos y] = \frac{d}{dx}[5x] - \frac{d}{dx}[4y]$$

For $\cos y$, apply the chain rule since y depends on x:

$$\cos x + \left(-\sin y \cdot \frac{dy}{dx}\right) = 5 - 4 \cdot \frac{dy}{dx}$$

$$\cos x - \sin y \cdot \frac{dy}{dx} = 5 - 4 \cdot \frac{dy}{dx}$$

Collect all $\frac{dy}{dx}$ terms on the left:

$$4 \cdot \frac{dy}{dx} - \sin y \cdot \frac{dy}{dx} = 5 - \cos x$$

$$(4 - \sin y)\frac{dy}{dx} = 5 - \cos x \implies \boxed{\frac{dy}{dx} = \frac{5 - \cos x}{4 - \sin y}}$$

(d) $x^3y^4 - y = x$

Differentiate both sides with respect to x:

$$\frac{d}{dx}[x^3y^4] - \frac{d}{dx}[y] = \frac{d}{dx}[x]$$

The first term requires the product rule, and y^4 requires the chain rule:

$$\frac{d}{dx}[x^3] \cdot y^4 + x^3 \cdot \frac{d}{dx}[y^4] - \frac{dy}{dx} = 1$$

$$3x^2 \cdot y^4 + x^3 \cdot 4y^3 \cdot \frac{dy}{dx} - \frac{dy}{dx} = 1$$

Collect all $\frac{dy}{dx}$ terms on the left:

$$4x^3y^3 \cdot \frac{dy}{dx} - \frac{dy}{dx} = 1 - 3x^2y^4$$

$$(4x^3y^3 - 1)\frac{dy}{dx} = 1 - 3x^2y^4$$

$$\boxed{\frac{dy}{dx} = \frac{1 - 3x^2y^4}{4x^3y^3 - 1}}$$

Example 3.27

A spherical balloon is being pumped up with air so that its volume changes at a rate of 1 ft^3 per second. How fast is the radius increasing when the diameter is 2 ft?

We are given:

$$\frac{dV}{dt} = 1\ \text{ft}^3/\text{sec}, \qquad d = 2\ \text{ft} \implies r = 1\ \text{ft}$$

The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Differentiate both sides with respect to t:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{d}{dr}\left[\frac{4}{3}\pi r^3\right] \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

Substitute $\frac{dV}{dt} = 1$ and $r = 1$:

$$1 = 4\pi(1)^2 \cdot \frac{dr}{dt} \implies 1 = 4\pi \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi} \approx \boxed{0.08\ \text{ft/sec}}$$

Related Rates

So far we used implicit differentiation to find $\frac{dy}{dx}$, how y changes relative to x. Now we apply the same idea to **related rates** problems, where instead of x and y we have quantities that both change over **time**.

The Big Idea!

When two or more quantities are related by an equation, their *rates of change* are also related. If we differentiate the equation with respect to time t, we get a relationship between the rates.

For example, if the radius r and volume V of a balloon are related by $V = \frac{4}{3}\pi r^3$, then their rates of change are related by:

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

So if we know how fast the volume is changing, we can find how fast the radius is changing - and vice versa.

Example 3.28

A 10-foot ladder leans against a wall. The base slides away from the wall at 2 ft/sec. At what rate is the top of the ladder sliding down the wall when the base is 8 feet from the wall?

$$\frac{dx}{dt} = 2 \text{ ft/sec}, \qquad \frac{dy}{dt} = ? \text{ when } x = 8 \text{ ft}$$

Step 1: Write an equation relating x and y.

By the Pythagorean theorem:

$$x^2 + y^2 = 10^2 = 100$$

Step 2: Find y when $x = 8$.

$$8^2 + y^2 = 100 \implies 64 + y^2 = 100 \implies y^2 = 36 \implies y = 6$$

Step 3: Differentiate both sides with respect to t.

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}[100]$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

Step 4: Plug in known values and solve.

$$2(8)(2) + 2(6) \cdot \frac{dy}{dt} = 0$$

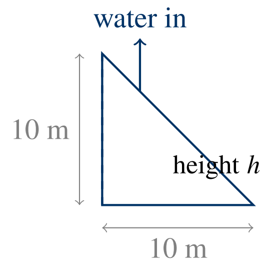
$$32 + 12 \cdot \frac{dy}{dt} = 0$$

$$12 \cdot \frac{dy}{dt} = -32 \implies \boxed{\frac{dy}{dt} \approx -2.67 \text{ ft/sec}}$$

The negative sign means the top of the ladder is sliding **down** the wall.

Example 3.29

A tank shaped like a prism on its side has a triangular base that is a right triangle of height 10 m and width 10 m. If the tank is being filled at 30 m^3/min, how quickly is the height changing?



$$\frac{dV}{dt} = 30 \text{ m}^3/\text{min}, \qquad \frac{dh}{dt} = ?$$

Step 1: Area of the triangular base.

$$\text{Area} = \frac{1}{2}(10)(10) = 50 \text{ m}^2$$

Step 2: The volume equation.

$$V = (\text{Area of base}) \cdot h = 50h$$

Step 3: Differentiate both sides with respect to t.

$$\frac{d}{dt}[V] = \frac{d}{dt}[50h]$$

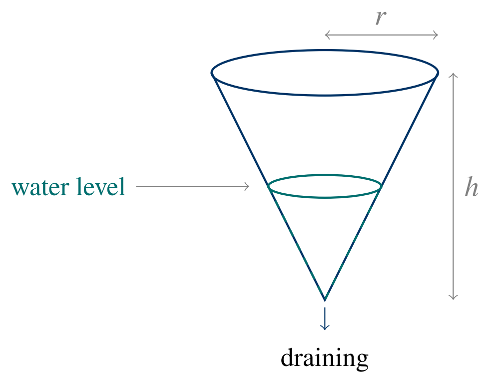
$$\frac{dV}{dt} = 50 \cdot \frac{dh}{dt}$$

Step 4: Plug in and solve.

$$30 = 50 \cdot \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{30}{50} = \boxed{0.6 \text{ m/min}}$$

Example 3.30

Water is draining from an inverted conical tank at 20 in^3/min. The tank is three times as tall as the radius at the top and holds 3000 in^3 of water. What is the rate of change of the radius when the height of the water is 12 in?



$$\frac{dV}{dt} = -20\ \text{in}^3/\text{min}, \qquad h = 3r \qquad \frac{dr}{dt} = ?\ \text{when}\ h = 12\ \text{in}$$

Step 1: The volume equation.

$$V = \frac{1}{3}\pi r^2 h$$

Since $h = 3r$, substitute h into the volume formula:

$$V = \frac{1}{3}\pi r^2(3r) = \pi r^3$$

Step 2: Find r when $h = 12$.

$$h = 3r \implies 12 = 3r \implies r = 4\ \text{in}$$

Step 3: Differentiate both sides with respect to t.

$$\frac{d}{dt}[V] = \frac{d}{dt}[\pi r^3]$$

$$\frac{dV}{dt} = \frac{d}{dr}[\pi r^3] \cdot \frac{dr}{dt} \implies \frac{dV}{dt} = 3\pi r^2 \cdot \frac{dr}{dt}$$

Step 4: Plug in known values and solve.

$$-20 = 3\pi(4)^2 \cdot \frac{dr}{dt} \implies -20 = 48\pi \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-20}{48\pi} \approx \boxed{-0.13\ \text{in/min}}$$

The negative sign means the radius is **decreasing** as the water drains.

3.6 L'Hôpital's Rule

Recall that when we try to evaluate a limit like

$$\lim_{x \to 0} \frac{\sin x}{x}$$

direct substitution gives $\frac{0}{0}$, an **indeterminate form**. This doesn't mean the limit is zero or undefined. It means we need a better method.

L'Hôpital's Rule gives us a good shortcut: when we get an indeterminate form, instead of doing algebra we can differentiate the numerator and denominator *separately* and try the limit again.

> ***Theorem 3.33:* L'Hôpital's Rule**
> If f and g are differentiable functions and
>
> $$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$
>
> then:
>
> $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$
>
> provided the limit on the right side exists.
>
> **Important warnings:**
>
> - L'Hôpital's rule only applies when substitution gives $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. **Always check the form first.**
> - You differentiate the numerator and denominator **separately** — this is NOT the quotient rule.
> - If after applying the rule you still get an indeterminate form, you can apply it **again**.

Let's revisit two limits that were hard before – now using L'Hôpital's rule.

(1) $\lim_{x \to 0} \frac{\sin x}{x}$

Direct substitution gives $\frac{0}{0}$, which means the L'Hôpital's rule applies.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \lim_{x \to 0} \cos x = \boxed{1}$$

(2) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Direct substitution gives $\frac{0}{0}$, so the L'Hôpital's rule applies here as well.

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{3x^2}{1} = \lim_{x \to 1} 3x^2 = \boxed{3}$$

Example 3.34

Determine whether L'Hôpital's rule applies to each limit.

(a) $\lim_{x\to 0} \frac{e^{2x}-1}{\cos x}$

Numerator $\to e^0 - 1 = 0$, Denominator $\to \cos(0) = 1$.

$$\text{Form: } \frac{0}{1} \quad \Longrightarrow \quad \text{L'Hôpital's rule does } \textbf{not} \text{ apply.}$$

Direct substitution works:

$$\lim_{x\to 0} \frac{e^{2x}-1}{\cos x} = \frac{0}{1} = \boxed{0}$$

(b) $\lim_{x\to 0} \frac{\cos x}{x}$

Numerator $\to \cos(0) = 1$, Denominator $\to 0$.

$$\text{Form: } \frac{1}{0} \quad \Longrightarrow \quad \text{L'Hôpital's rule does } \textbf{not} \text{ apply.}$$

$$\boxed{\text{DNE}}$$

(c) $\lim_{x\to 2} \frac{x^2-4}{2x-6}$

Numerator $\to 4 - 4 = 0$, Denominator $\to 4 - 6 = -2$.

$$\text{Form: } \frac{0}{-2} \quad \Longrightarrow \quad \text{L'Hôpital's rule does } \textbf{not} \text{ apply.}$$

Direct substitution works:

$$\lim_{x\to 2} \frac{x^2-4}{2x-6} = \frac{0}{-2} = \boxed{0}$$

(d) $\lim_{x\to 1} \frac{x^4-1}{x^2-1}$

Numerator $\to 1 - 1 = 0$, Denominator $\to 1 - 1 = 0$.

$$\text{Form: } \frac{0}{0} \quad \Longrightarrow \quad \text{L'Hôpital's rule } \textbf{applies}.$$

Differentiate numerator and denominator separately:

$$\lim_{x\to 1} \frac{x^4-1}{x^2-1} = \lim_{x\to 1} \frac{4x^3}{2x} = \lim_{x\to 1} 2x^2 = \boxed{2}$$

Example 3.35: Compute the following limits.

(a) $\lim_{x\to 2} \frac{\ln(x/2)}{x-2}$

Check: numerator $\to \ln(1) = 0$, denominator $\to 0$. Form: $\frac{0}{0}$ —> rule applies.

$$\lim_{x\to 2} \frac{\ln(x/2)}{x-2} = \lim_{x\to 2} \frac{\frac{1}{x}}{1} = \lim_{x\to 2} \frac{1}{x} = \frac{1}{2} \implies \boxed{\frac{1}{2}}$$

(b) $\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3}$

Check: numerator $\to 0$, denominator $\to 0$. Form: $\frac{0}{0}$ —> rule applies.

$$\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3} = \lim_{\theta\to 0} \frac{1-\cos\theta}{3\theta^2}$$

Check again: numerator $\to 1 - 1 = 0$, denominator $\to 0$. Still $\frac{0}{0}$ —> apply again.

$$= \lim_{\theta\to 0} \frac{\sin\theta}{6\theta}$$

Check again: numerator $\to 0$, denominator $\to 0$. Still $\frac{0}{0}$ —> apply again.

$$= \lim_{\theta\to 0} \frac{\cos\theta}{6} = \frac{\cos(0)}{6} = \frac{1}{6} \implies \boxed{\frac{1}{6}}$$

(c) $\lim_{t\to -3} \frac{t^3 - 4t + 15}{t^2 - t - 12}$

Check: numerator $\to -27 + 12 + 15 = 0$, denominator $\to 9 + 3 - 12 = 0$. Form: $\frac{0}{0}$ —> rule applies.

$$\lim_{t\to -3} \frac{t^3 - 4t + 15}{t^2 - t - 12} = \lim_{t\to -3} \frac{3t^2 - 4}{2t - 1} = \frac{3(-3)^2 - 4}{2(-3) - 1} = \frac{27-4}{-6-1} = \frac{23}{-7} \implies \boxed{-\frac{23}{7}}$$

(d) $\lim_{x\to \pi} \frac{\sin x}{1 + \cos x}$

Check: numerator $\to \sin(\pi) = 0$, denominator $\to 1 + \cos(\pi) = 1 - 1 = 0$. Form: $\frac{0}{0}$ —> rule applies.

$$\lim_{x\to \pi} \frac{\sin x}{1+\cos x} = \lim_{x\to \pi} \frac{\cos x}{-\sin x} = \frac{\cos(\pi)}{-\sin(\pi)} = \frac{-1}{0}$$

The denominator goes to 0 while the numerator goes to -1, so:

$$\boxed{\text{DNE}}$$

Theorem 3.36: **L'Hôpital's Rule (Extended)**

L'Hôpital's rule also works for limits involving infinity. Let f and g be differentiable functions and let a be any real number or $\pm\infty$. If:

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right side exists.

In short, L'Hôpital's rule applies to both $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$ indeterminate forms. ***Always check the form first before applying the rule.***

Example 3.37

Compute the following limits.

(a) $\lim_{x\to 0^+} \frac{\sin 2x}{\sqrt{5x}}$

Check: numerator $\to 0$, denominator $\to 0$. Form: $\frac{0}{0}$ —> rule applies.

$$\lim_{x\to 0^+} \frac{\sin 2x}{\sqrt{5x}} = \lim_{x\to 0^+} \frac{2\cos(2x)}{\frac{1}{2}(5x)^{-1/2}\cdot 5} = \lim_{x\to 0^+} \frac{2\cos(2x)}{\frac{5}{2\sqrt{5x}}}$$

$$= \lim_{x\to 0^+} 2\cos(2x)\cdot\frac{2\sqrt{5x}}{5} = \lim_{x\to 0^+} \frac{4\sqrt{5x}\cos(2x)}{5} = \frac{4\sqrt{0}\cdot\cos(0)}{5} = \boxed{0}$$

(b) $\lim_{x\to -\infty} \frac{e^x - 1}{x^2 + x}$

Check: numerator $\to e^{-\infty} - 1 = 0 - 1 = -1$, denominator $\to \infty$. Form: $\frac{-1}{\infty}$ —> L'Hôpital's rule does **not** apply.

Direct substitution:

$$\lim_{x\to -\infty} \frac{e^x - 1}{x^2 + x} = \frac{-1}{\infty} = \boxed{0}$$

(c) $\lim_{x\to \infty} \frac{x - 8x^2}{12x^2 + 5x}$

Check: numerator $\to -\infty$, denominator $\to \infty$. Form: $\frac{-\infty}{\infty}$ —> rule applies.

$$\lim_{x\to\infty} \frac{x - 8x^2}{12x^2 + 5x} = \lim_{x\to\infty} \frac{1 - 16x}{24x + 5}$$

Check again: numerator $\to -\infty$, denominator $\to \infty$. Still $\frac{-\infty}{\infty}$ —> apply again.

$$= \lim_{x\to\infty} \frac{-16}{24} = \boxed{-\frac{2}{3}}$$

(d) $\lim_{\theta\to\pi/2} \frac{\sec\theta}{\tan\theta}$

Check: numerator $\to$ undefined, denominator $\to$ undefined.

L'Hôpital's rule does **not** apply directly. Instead, rewrite using trig identities:

$$\frac{\sec\theta}{\tan\theta} = \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} = \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} = \frac{1}{\sin\theta}$$

Now take the limit:

$$\lim_{\theta\to\pi/2} \frac{1}{\sin\theta} = \frac{1}{\sin(\pi/2)} = \frac{1}{1} = \boxed{1}$$

Example 3.38

Determine $\lim_{x\to 0} x \ln x$.

Direct substitution gives $0 \cdot (-\infty)$ —> this is an indeterminate form, but it is **not** in the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ so we cannot apply L'Hôpital's rule yet.

So, we rewrite $x \ln x$ as:

$$x \ln x = \frac{\ln x}{\frac{1}{x}}$$

Check:

Numerator $\to -\infty$

Denominator $\to +\infty$. Form: $\frac{-\infty}{+\infty}$ —> rule applies.

$$\lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

Simplify:

$$= \lim_{x\to 0} \frac{1}{x} \cdot \frac{x^2}{-1} = \lim_{x\to 0} \frac{x^2}{-x} = \lim_{x\to 0}(-x) = \boxed{0}$$

Example 3.39

Determine $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

Direct substitution gives $\infty - \infty$ —> an indeterminate form. We cannot apply L'Hôpital's rule yet.

So, we first combine the two fractions into one.

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x \cdot \ln x - (x-1)}{(x-1)\ln x} = \frac{x \ln x - (x-1)}{(x-1)\ln x}$$

Check:

Numerator $\rightarrow 1 \cdot \ln 1 - 0 = 0$

Denominator $\rightarrow 0 \cdot \ln 1 = 0$. Form: $\frac{0}{0}$ —> rule applies.

$$\lim_{x\to 1} \frac{x \ln x - (x-1)}{(x-1)\ln x} = \lim_{x\to 1} \frac{(1)\ln x + x \cdot \frac{1}{x} - 1}{\left(\frac{1}{x}\right)(x-1) + \ln x \cdot (1)}$$

$$= \lim_{x\to 1} \frac{\ln x + 1 - 1}{\frac{x-1}{x} + \ln x} = \lim_{x\to 1} \frac{\ln x}{\frac{x-1}{x} + \ln x}$$

Check:

Numerator $\rightarrow \ln 1 = 0$

Denominator $\rightarrow 0 + 0 = 0$. Still $\frac{0}{0}$ —> apply again.

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{x \cdot 1 - (x-1) \cdot 1}{x^2} + \frac{1}{x}} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}}$$

Substitute $x = 1$:

$$= \frac{\frac{1}{1}}{\frac{1}{1} + \frac{1}{1}} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

The examples and solutions in these notes are based on lectures and class sessions given by **Dr. Rachelle Bouchat** in Calculus I (MAT 135) at Berea College, Spring 2026.
